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# Gravitational Ultrarelativistic Interaction Of Classical Particles In The Context Of Unification Of Interactions

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GRAVITATIONAL ULTRARELATIVISTIC INTERACTION  
OF CLASSICAL PARTICLES IN THE CONTEXT  
OF UNIFICATION OF INTERACTIONS

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**Abstract**

The dependence of the gravitoelectric and gravitomagnetic field components on the relative velocity of a Schwarzschild source and a local observer is considered. Three kinds of these components are identified at the ultrarelativistic velocity, namely, which are proportional to  $\gamma^2$ ,  $\gamma$ , and which are independent of  $\gamma$  ( $\gamma$  is the relativistic Lorentz factor). The physical situations which evince the roles of different components are described. Particularly the reaction of spin on the ultrarelativistic gravitomagnetic field is analysed. A tendency of gravitational and electromagnetic interactions to approach in quantitative terms at ultrarelativistic velocities is discussed.

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## 1 Introduction

The theory of unification of interactions has as its aim to elucidate the basic properties of the micro-world at high energies. An interesting question is the following: is there not a tendency of gravitational and electromagnetic interactions to approach (at least in quantitative terms) already at a macro-level in situations where the interacting objects have very high relative velocities? In other words, are we not getting an indication, in the framework of general theory of relativity and classical electromagnetism, of a tendency of diminution of the difference between these interactions when relative velocities of the interacting classical particles are ultrarelativistic?

Certainly, investigations of analogies between gravitational and electromagnetic interactions have their own long history on the level of Newton's Law of universal gravitation and Coulomb's Law. In the last decades fundamentally new common traits of these interactions were brought to light. In particular, terms such as "gravitoelectric field", "gravitomagnetic field", and "gravitoelectromagnetism" have gained currency within the general theory of relativity, along with the elaboration of their context, [1]. Particularly worthy of attention is an early publication [2].

Interesting aspects of analogies between gravitation and electromagnetism are revealed when one looks at the behavior of a classical test particle with spin in a gravitational field [3]. Examining the Mathisson-Papapetrou (MP) equations [4] in a Kerr field, [3] considers gravitational spin-orbit and spin-spin interactions in respective approximations in power of  $1/c$  and compares them with analogous electromagnetic interactions.

The concept of the electromagnetic field was preceded by concepts of two independent entities, the electric and the magnetic interaction, which became unified in Maxwell's theory. On the other hand, in the case of Einstein's theory of gravitation just the opposite took place: the general theory of relativity was from the start a theory of a single gravitational field and only with time did many investigators begin to feel the need to treat as separate (although closely linked) two of its components: gravitoelectric and gravitomagnetic. The separation of gravitoelectric and gravitomagnetic components of the gravitational field in the general theory of relativity is carried out with the Riemann tensor as the basic characteristic of the field [2, 5]. In the local orthonormal basis, the gravitoelectric components of the gravitational field  $E_{(k)}^{(i)}$  are determined by the relationship

$$E_{(k)}^{(i)} = R^{(i)(4)}_{(k)(4)}, \quad (1)$$

where  $R^{(i)(4)}_{(k)(4)}$  denotes local components of the Riemann tensor. (Here and in the following the indices of the orthogonal tetrads are placed in round parentheses; Latin indices run through values 1, 2, 3, while Greek indices through 1, 2, 3, 4). Correspondingly, for gravitomagnetic components  $B_{(k)}^{(i)}$  we have

$$B_{(k)}^{(i)} = -\frac{1}{2}R^{(i)(4)}_{(m)(n)}\varepsilon^{(m)(n)}_{(k)}, \quad (2)$$

where  $\varepsilon^{(m)(n)}_{(k)}$  is the Levi-Civita tensor.

We shall begin our investigation by analyzing relationships (1) and (2) in the concrete case of a gravitational field created by a Schwarzschild mass moving relative to an observer with arbitrary velocity.

We point out that in interesting paper [6] the similar to a certain extent problem was considered. Namely, "...it is shown that the gravitational field of a fast-moving mass bears an increasing resemblance to a plane gravitational wave, the greater the speed of the mass" [6], p. 96. However, in this paper the gravitational field of a fast-moving Schwarzschild mass was considered only in the context of investigations of the gravitational waves in the general theory of relativity. The influence of components (1), (2) on the other masses was not under investigation in [6].

The gravitational field of a massless particle which moves with the velocity of light was considered in [7]. It was shown that the gravitational field of this particle is nonvanishing only on a plane containing the particle and orthogonal to the direction of motion. The results of [7] were used in [8] for investigating the ultrarelativistic collision of two black holes. The detail elucidation of this problem one can find in [9]. Passing from the infinite Lorentz  $\gamma$ -factor (the case of a massless particle with the velocity of light) to the large but finite  $\gamma$  (a massive ultrarelativistic particle) is described taking into account the small parameter  $\gamma^{-1}$  and the corresponding small corrections to the metric of a massless particle [9]. This approach to the description of the dependence of the gravitational field of a moving massive particle on the  $\gamma$  differs from the method and results of [6]. In [6] the Riemann tensor components were calculated for any value  $1 < \gamma < \infty$  without the consideration of the case of the infinite  $\gamma$  as the initial approximation.

In what follows, we shall use a system of units where  $c = G = 1$ .

## **2 Dependence of the gravitoelectromagnetic field of a moving Schwarzschild source on the Lorentz $\gamma$ -factor**

The results of this Section may be considered as a direct development of [6]. The main idea of [6] is the comparison of the canonical forms of the Riemann tensor for a plane gravitational wave and a fast-moving mass (the  $3 \times 3$  matrices  $P$  and  $Q$  in the notation of [6]). Our purpose is the analysis of the action of different components (1), (2), as measured by a fast-moving

observer in the Schwarzschild field, on the test masses.

We shall label the reference frame, which moves with respect to the source of the Schwarzschild field in an arbitrary direction and with arbitrary velocity, by a set of corresponding tetrads  $\lambda_{(\beta)}^\alpha$ . The Schwarzschild metric we consider in standard coordinates  $x^1 = r, x^2 = \theta, x^3 = \varphi, x^4 = t$ . For expediency and without loss of generality we assume the directions of the space axes of the ortho-reference to be as follows: The first axis is perpendicular to the plane determined by the direction of observer motion and the radial direction to the field source. (Obviously, in the particular case of radial motion there is freedom of choice). The second axis coincides with the direction of motion. As a consequence, we note that the following tetrad components have zero components:  $\lambda_{(1)}^1, \lambda_{(1)}^3, \lambda_{(2)}^2, \lambda_{(3)}^2$ . For evaluation of other components we shall use the general relationship between tetrad components and the metric tensor  $g^{\alpha\pi}$ :

$$\lambda_{(\beta)}^\alpha \lambda_{(\rho)}^\pi \eta^{(\beta)(\rho)} = g^{\alpha\pi}, \quad (3)$$

where  $\eta^{(\beta)(\rho)} = \text{diag}(-1, -1, -1, 1)$  is the Minkowski tensor.

For the Schwarzschild metric, where only the diagonal elements of the tensor  $g^{\alpha\pi}$  are different from zero, the system of the ten algebraic equations involving tetrad components of (3) may be separated into subsystems of lower dimensionality, permitting to determine all components that are different from zero:

$$\begin{aligned} \lambda_{(1)}^2 &= \sqrt{-g^{22}}, & \lambda_{(2)}^1 &= u^1 u^4 \sqrt{\frac{g_{44}}{u_4 u^4 - 1}}, & \lambda_{(2)}^3 &= u^3 u^4 \sqrt{\frac{g_{44}}{u_4 u^4 - 1}}, \\ \lambda_{(2)}^4 &= \sqrt{\frac{u_4 u^4 - 1}{g_{44}}}, & \lambda_{(3)}^1 &= u^3 \sqrt{\frac{g^{11} g_{33}}{u_4 u^4 - 1}}, & \lambda_{(3)}^3 &= -u^1 \sqrt{\frac{g^{33} g_{11}}{u_4 u^4 - 1}}, \\ \lambda_{(4)}^1 &= u^1, & \lambda_{(4)}^3 &= u^3, & \lambda_{(4)}^4 &= u^4, \end{aligned} \quad (4)$$

where  $u^\mu$  is the 4-vector of the observer velocity. (The  $\theta$  angle is measured such that the observer moves in the plane  $\theta = \pi/2$ , which entails  $u^2 = 0$ . Expressions (4) were used in [7] while dealing with another problem).

According to (1) and (2), in order to evaluate  $E_{(k)}^{(i)}$  and  $B_{(k)}^{(i)}$  it is necessary to have the values of the local components of the Riemann tensor, which are connected to its global components by the well known relation

$$R_{(\alpha)(\beta)(\gamma)(\delta)} = \lambda_{(\alpha)}^\mu \lambda_{(\beta)}^\nu \lambda_{(\gamma)}^\rho \lambda_{(\delta)}^\sigma R_{\mu\nu\rho\sigma}. \quad (5)$$

Non-zero components of the Riemann tensor, expressed in standard Schwarzschild coordinates for  $\theta = \pi/2$ , are

$$\begin{aligned} R_{1212} = R_{1313} &= \frac{m}{r-2m}, & R_{2323} &= -2mr, \\ R_{1414} &= \frac{2m}{r^3}, & R_{2424} = R_{3434} &= -\frac{m}{r} \left(1 - \frac{2m}{r}\right). \end{aligned} \quad (6)$$

Using (4)–(6), we find the following non-zero components of the Riemann tensor that are present in (1) and (2):

$$\begin{aligned} R^{(1)(4)}_{(1)(4)} &= -\frac{m}{r^3}(3u_3u^3 - 1), & R^{(2)(4)}_{(2)(4)} &= -\frac{2m}{r^3} \frac{u^1u^1}{g_{44}(u_4u^4 - 1)} \\ &+ \frac{m}{r} \frac{u^3u^3}{u_4u^4 - 1}, & R^{(2)(4)}_{(3)(4)} = R^{(3)(4)}_{(2)(4)} &= -\frac{3m}{r^2} \frac{u^1u^3u^4}{u_4u^4 - 1}, \\ R^{(3)(4)}_{(3)(4)} &= -\frac{m}{r^3}(u_4u^4 - 1) + \frac{2m}{r^3} u_4u^4 \frac{u_3u^3 - u_1u^1}{u_4u^4 - 1}, \\ R^{(1)(4)}_{(1)(2)} &= \frac{3mu^3u^3u^4}{r\sqrt{u_4u^4 - 1}} \left(1 - \frac{2m}{r}\right)^{1/2}, \\ R^{(1)(4)}_{(1)(3)} &= -\frac{3mu^1u^3}{r^2\sqrt{u_4u^4 - 1}} \left(1 - \frac{2m}{r}\right)^{-1/2}, \\ R^{(2)(4)}_{(2)(3)} &= \frac{3mu^1u^3}{r^2\sqrt{u_4u^4 - 1}} \left(1 - \frac{2m}{r}\right)^{-1/2}, \\ R^{(3)(4)}_{(2)(3)} &= \frac{3mu^3u^3u^4}{r\sqrt{u_4u^4 - 1}} \left(1 - \frac{2m}{r}\right)^{1/2}. \end{aligned} \quad (7)$$

Using (7) in (1), we obtain the following non-zero components of the gravitoelectric field:

$$\begin{aligned} E^{(1)}_{(1)} &= \frac{m}{r^3}(1 + 3u_\perp^2), & E^{(2)}_{(2)} &= -\frac{2m}{r^3} + \frac{3m}{r^3} \frac{u_\perp^2}{u_4u^4 - 1}, \\ E^{(2)}_{(3)} = E^{(3)}_{(2)} &= -\frac{3m}{r^3} \frac{u_\parallel u_\perp u^4}{u_4u^4 - 1}, & E^{(3)}_{(3)} &= \frac{m}{r^3} - \frac{3m}{r^3} \frac{u_\perp^2 u_4u^4}{u_4u^4 - 1}. \end{aligned} \quad (8)$$

where  $u_{\parallel} = u^1$  is the radial component of the 4-velocity,  $u_{\perp} = ru^3$  is its tangential component. Because of the condition  $u_{\mu}u^{\mu} = 1$ , here we have the following relationship:

$$u_4u^4 - 1 = u_{\perp}^2 + \left(1 - \frac{2m}{r}\right)^{-1} u_{\parallel}^2. \quad (9)$$

Similarly, using (7) in (2), we obtain the non-zero components of the gravitomagnetic field,

$$B_{(2)}^{(1)} = B_{(1)}^{(2)} = \frac{3mu_{\parallel}u_{\perp}}{r^3\sqrt{u_4u^4 - 1}} \left(1 - \frac{2m}{r}\right)^{-1/2},$$

$$B_{(3)}^{(1)} = B_{(1)}^{(3)} = \frac{3mu_{\perp}^2u^4}{r^3\sqrt{u_4u^4 - 1}} \left(1 - \frac{2m}{r}\right)^{1/2}. \quad (10)$$

Let us stress that relationships (8) and (10) hold true for any arbitrary velocity of the observer.

We shall begin the examination of the components of (8) by simply noting that they have non-zero values even in the Newtonian limit, when  $|u_{\parallel}| \ll 1$ ,  $|u_{\perp}| \ll 1$ ,  $u^4 \approx 1$ . This had to be expected, inasmuch as in the Newtonian theory there is correspondence between the  $E_{(k)}^{(i)}$  components and the so-called tidal matrix  $E_{ij}$ , where

$$E_{ij} = -\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \varphi(\vec{x}, t), \quad (11)$$

i.e. the second derivatives of the Newtonian potential [8]. The denomination  $E_{ij}$  is not fortuitous in view of the fact that the components of the tidal acceleration  $a_{i_{tidal}}$  in the Newtonian theory are, see [8],

$$a_{i_{tidal}} = E_{ij}r_j. \quad (12)$$

In the general theory of relativity the components  $E_{(k)}^{(i)}$  are also linked with the tidal acceleration, more exactly with the equation of deviation of geodesic lines. Taking (1) into account, this may be written as

$$\frac{D^2 l^{(i)}}{ds^2} = E_{(k)}^{(i)} l^{(k)}, \quad (13)$$

where  $s$  is the proper time,  $l^{(i)}$  is the vector of relative deviation of two neighboring geodesic lines. It is the equation of deviation of geodesics that

was used in [9], Sec. 31.2, for the analysis of tidal forces felt by an observer while falling onto a Schwarzschild black hole.

According to (13) we have

$$a_{tidal}^{(i)} = E_{(k)}^{(i)} l^{(k)}, \quad (14)$$

Let us note that in [9], Sec. 31.2, the components  $E_{(k)}^{(i)}$  are not explicitly mentioned in the analysis of tidal force, but only Riemann tensor components which, according to (1), correspond to  $E_{(k)}^{(i)}$ . (At another place in [9], in Sec. 1.6, there is mention of the analogy between one part of the Riemann tensor components and the electric field components, and between the other part and the magnetic field components, but the relationships (1) and (2) are not given explicitly). Let us also note that in [9], Sec. 31.2, the analysis of tidal forces is limited to the case of radial motion, when  $u_{\perp} = 0$ . For such motion, according to (8), the components  $E_{(k)}^{(i)}$  assume the following values:

$$E_{(1)}^{(1)} = \frac{m}{r^3}, \quad E_{(2)}^{(2)} = -\frac{2m}{r^3}, \quad E_{(3)}^{(2)} = E_{(2)}^{(3)} = 0, \quad E_{(3)}^{(3)} = \frac{m}{r^3}, \quad (15)$$

that is, they appear completely independent of  $u_{\parallel}$ . In view of (14), in such a case the tidal acceleration also does not depend on the velocity of radial motion. The fact that in a radial fall of the observer the tidal forces felt by him are independent of this velocity is, in essence, noted in [9], Sec. 31.2, (while noting at the same time the analogy with electromagnetism). As a consequence, it is stated in [9] that for a radial falling observer the tidal forces increase sharply only at  $r \rightarrow 0$ . The question which remained unanswered in [9] was the following: what will change if the fall is non-radial? In view of expressions (8), this question can be readily answered. According to (8), the expressions giving the components of the gravitoelectric field in the case of non-radial motion differ significantly from expressions (15) only when the velocity becomes ultrarelativistic. Indeed, inasmuch as  $u_{\parallel}$ ,  $u_{\perp}$ ,  $u^4$  are proportional to the Lorentz relativistic  $\gamma$ -factor, (8) gives us, for  $|u_{\perp}| \gg 1$ ,  $|u_{\parallel}| \gg 1$ ,  $u^4 \gg 1$

$$E_{(1)}^{(1)} \approx \frac{3m}{r^3} \gamma^2, \quad E_{(2)}^{(2)} \approx \frac{3m}{r^3}, \quad E_{(3)}^{(2)} = E_{(2)}^{(3)} \sim \frac{3m}{r^3} \gamma, \quad E_{(3)}^{(3)} \sim \frac{3m}{r^3} \gamma^2. \quad (16)$$

Comparing (16) with (15), we see that while the components in (15) assume arbitrarily large values only when  $r \rightarrow 0$ , the components  $E_{(1)}^{(1)}$ ,  $E_{(3)}^{(2)} = E_{(2)}^{(3)}$ ,



$E_{(3)}^{(3)}$  of (16) become arbitrarily large already at finite values of  $r$ , provided  $\gamma \rightarrow \infty$ . (Here we leave aside the question of how to impart to an observer, in practice, a velocity corresponding to large values of  $\gamma$ ). Thus, according to (14), an observer in a Schwarzschild field runs the risk of being torn apart by tidal forces not only at  $r \rightarrow 0$ , i.e. under the surface of the horizon (which is described in [9], Sec. 31.2), but even at large values of  $r$  if his velocity becomes ultrarelativistic.

Obviously, the expressions for the components of the gravitoelectric field (8) and (16) are independently valid, without having to be linked with equations of deviation of geodesic lines and tidal forces. The significance of (8) and (16) resides primarily in characterizing the gravitational field created by a moving Schwarzschild source.

Let us now look at the components of the gravitomagnetic field, (10). It is easy to see that components (10) are different from zero only when  $u_{\perp} \neq 0$ , that is only when the observer is moving non-radially. (As is well known, a similar situation arises in electrodynamics for components of the vector of magnetic field intensity of a moving electric charge). Quite generally, the values of components (10) depend significantly on observer motion. In the low relativistic region, with  $|u_{\perp}| \ll 1$ ,  $|u_{\parallel}| \ll 1$ ,  $u^4 \approx 1$ , the common multiplier  $m/r^3$  of components is further multiplied by corresponding small factors. Whereas in the ultrarelativistic region, where  $|u_{\perp}| \gg 1$ ,  $|u_{\parallel}| \gg 1$ ,  $u^4 \gg 1$ , this multiplier is further multiplied by large factors, because in this case, according to (10), we have:

$$B_{(2)}^{(1)} = B_{(1)}^{(2)} \sim \frac{3m}{r^3} \gamma, \quad B_{(3)}^{(1)} = B_{(1)}^{(3)} \sim \frac{3m}{r^3} \gamma^2. \quad (17)$$

As we have seen, in the Newtonian limit the components of the gravitomagnetic field (10) have zero values, in contrast to the gravitoelectric field. Moreover, at low relativistic velocities the absolute values of components (10) are considerably smaller than the components (8). Yet at ultrarelativistic velocities, the largest components  $B_{(k)}^{(i)}$  from (17) and  $E_{(k)}^{(i)}$  from (16) are of the same order of magnitude, determined by the factor  $3m\gamma^2/r^3$ . (We point out that this factor is present in the expression for the amplitude of the gravitational wave from [6]). Consequently, we can conclude that the two aspects of the single gravitational interaction, the gravitoelectric and the gravitomagnetic, show, in the ultrarelativistic range, a tendency of qualitatively drawing together, even though at low velocities they differ substantially.

Before comparing the gravitational interaction with the electromagnetic in the ultrarelativistic range, we shall examine an important physical situation which evinces the role of the gravitomagnetic interaction.

### 3 A classical particle with spin in an ultra-relativistic gravitational field

Relationships (13), (14) describe a simple experiment in which an observer moving in a Schwarzschild field can determine the values of components of the gravitoelectric field in his own frame of reference. The question arises which experiment would permit this observer to determine the components of the gravitomagnetic field. As is shown in [7], such an experiment can be carried out by observing the behavior of a test particle with spin. Indeed, according to (9) in [7], the local components of the 3-acceleration  $a_{(i)}$  with which the test particle with spin deviates from free geodesic fall in an arbitrary gravitational field is given by the relationship

$$a_{(i)} = -\frac{s_{(1)}}{M} R_{(i)(4)(2)(3)}, \quad (18)$$

where  $M$  is the mass of a test particle. (Here the space axes of the reference frame are chosen such that the spin points along the first axis, so that the spin components are  $s_{(2)} = s_{(3)} = 0$ . Expression (18) is a direct consequence of the MP equations). Taking (2) into consideration, it is not difficult to see that the right side of (18) contains the components of the gravitomagnetic field.

For our concrete case of observer motion in a Schwarzschild field, characterized by the set of tetrads (4), we have

$$a_{(i)} = \frac{s_{(1)}}{M} B_{(i)}^{(1)}. \quad (19)$$

This result is obtained from (18) taking into account the appropriate components of the curvature tensor from (7) and expressions (10). The non-zero values of  $B_{(k)}^{(i)}$  in (18) come from (10). Even though the right sides of relationships (19) and (14) have a similar appearance, what is essential is that they contain different components of the gravitational field: in (19) gravitomagnetic and in (14) gravitoelectric. (Let us stress that we are dealing with one and the same reference frame, one connected with an observer moving

in a Schwarzschild field). Correspondingly, the nature of forces that cause accelerations (14) and (19) is different: in case of (14) these are tidal forces, and in (19) it is the spin-orbit force. (Detailed discussion of this question may be found in [7]). Referring to (10) we obtain the magnitude of the 3-acceleration  $|\vec{a}|$  with components (19):

$$|\vec{a}| = \frac{3m}{r^3} \frac{|s_{(1)}u_{\perp}|}{M} \sqrt{1 + u_{\perp}^2}. \quad (20)$$

According to (17), the acceleration components (19) depend, in the case of ultrarelativistic non-radial motion, on the Lorentz  $\gamma$ -factor such that  $a_{(2)} \sim \gamma$ ,  $a_{(3)} \sim \gamma^2$ . The component  $a_{(1)}$  remains equal to zero at any velocity in the case at hand, where the spin is directed along the first spacial vector of the reference frame, which direction is perpendicular to the plane determined by the direction of observer motion and the radial direction. This is so because in this case the corresponding component of the gravitational field is zero also. Expression (20) also shows that  $|\vec{a}| \sim \gamma^2$ .

Thus, the fact that at ultrarelativistic velocity the largest component of the gravitomagnetic field (17) is proportional to  $\gamma^2$  entails, on account of (19), that the spin-orbit acceleration is also proportional to  $\gamma^2$ .

Both, the MP equations and relationships (18)–(20) which follow from them, are rigorously valid for the model of a point test particle with spin, with tidal forces not coming into play. Certainly, for any real macroscopic test particle with rotational motion, tidal and spin-orbit forces become important.

Together, relationships (14) and (19) permit to evaluate these forces. The most significant conclusion of these evaluations lies in the result that for ultrarelativistic non-radial motions the values of these forces in the proper frame of the particle are both proportional to  $\gamma^2$ .

## 4 Comparison of gravitational and electromagnetic interactions of two particles moving with an ultrarelativistic relative velocity

We shall examine two situations where two particles are mutually interacting. In the first case, we consider two electrically neutral particles with the mass

of one being considerably greater than the mass of the other. The particle with smaller mass is endowed with classical spin (internal angular momentum). Thus, we can consider this particle to be the test particle with spin, moving in the gravitational field of the more massive particle, which in its own frame is described by a Schwarzschild metric.

In the second case, the particles carry electric charge, with one charge being considerably larger than the other. Moreover, the particle with the smaller charge has a magnetic moment arising from its internal rotation. The masses of these particles are such that at low relative velocities, when the Coulomb Law and Newton's Law of gravitational attraction hold true, the force of the electric interaction is very much larger than the gravitational attraction. Again, the particle with the smaller charge and the magnetic moment may be regarded as the test particle. Thus, we may consider that the first pair of particles interacts only gravitationally, and the second pair only electrically.

Let us inquire, in the two cases, how the forces resulting from the gravitational and the electromagnetic interactions, respectively, depend on the magnitude of the relative velocities of the particles. We shall assume that the particles are sufficiently distant one from the other to be able to neglect the respective gravitational and electromagnetic radiation. In accordance with the analysis carried out in the preceding sections, in the first case the force is due to spin and it increases with increasing relative velocity proportionally to  $\gamma^2$ , as long as the test mass is not moving radially, i.e. it is not moving along the line joining the two masses.

At the same time, in the second case, classical electrodynamics tells us that the force acting on the test particle with the magnetic field is proportional to  $\gamma$ . This means that, no matter how small the gravitational interaction may be in comparison with the electromagnetic interaction in the subrelativistic range of velocities, in passing into the ultrarelativistic range the ratio of the respective forces could, in principle, change to an extent of both forces becoming of the same order of magnitude, provided  $\gamma$  becomes large enough.

Similar conclusions may be drawn in a third situation, where a model proton interacts with a model electron. (Here we consider two classical particles with masses and charges of a proton and an electron, respectively. In this case, for the description of the gravitational field of the proton as the more massive particle, one has to refer to the Reissner-Nordstrom metric, rather the Schwarzschild metric, but this does not change the conclusion in

principle).

In the Introduction we asked the question, in the framework of general relativity and classical electrodynamics, if there is a tendency of gravitational and electromagnetic interactions to approach quantitatively with increasing relative velocity of interacting particles. We have shown that this question may be answered in the affirmative.

## 5 A case of the ultrarelativistic motion of a classical spinning particle in a Schwarzschild field and the corresponding solution of the Dirac equation

The physical measurements with the ultrarelativistic macroscopic masses are, at least at present, unattainable. Nonetheless, the results from Sections 3 and 4 are not merely academic. If only because the known fact that the general covariant Dirac equation passes, in a quasiclassical approximation, into the MP equations. A concrete problem that should be tackled, is obtaining solutions of the general covariant Dirac equation in a Schwarzschild field corresponding to ultrarelativistic electrons. It is not difficult to check that the Mathisson-Papapetrou equations in a Schwarzschild field have a strict partial solution describing the circular motion of a spinning test particle around the field source on the orbit with  $r = 3m$ . The relationship between the components of the particle's 4-velocity  $u_{\perp} \equiv r\dot{\varphi}$  and the 3-vector spin component  $S_2 \equiv S_{\theta}$  is

$$u_{\perp} = -\frac{3mM}{S_{\theta}} \quad (21)$$

(as above, here we use the standard Schwarzschild coordinates; spin is perpendicular to the plane of motion  $\theta = \pi/2$ , therefore  $S_1 = 0$ ,  $S_3 = 0$ ). It is necessary to take into account the condition for a spinning test particle  $|S_0|/Mr \ll 1$  where  $|S_0|$  is the value of the spin of a test particle as measured by the comoving observer [3] (in our case there is the relation  $|S_{\theta}| = ru_4|S_0|$ ). Therefore, for the value  $|u_{\perp}|$  from (21) we have  $|u_{\perp}| \gg 1$ , i.e. for the motion on the circular orbit with  $r = 3m$  a particle must possess the ultrarelativistic velocity, the higher the spin is smaller. Formally, at  $S_{\theta} = 0$  the value  $|u_{\perp}|$  in (21) must be infinitely large, that is the particle must move with the speed of

light. This fact corresponds to the known result following from the geodesic equations in a Schwarzschild field: the circular nonisotropic geodesic orbits exist only at  $r > 3m$  and, formally, for the motion on the orbit  $r = 3m$  a test particle without spin must possess the speed of light. In practice it means that only the beam of light can move on the orbit with  $r = 3m$ .

So, according to the MP Eqs. the spin of a test particle allows its ultrarelativistic motion on the circular orbit  $r = 3m$ . The calculations of the gravitational spin-orbit acceleration on the orbit  $r = 3m$  according to (20), (21) give

$$|\vec{a}| = \frac{\sqrt{3}}{9m}. \quad (22)$$

For the quantitative comparison, we point out that value (22) is close to the Newtonian value of the free fall acceleration for the mass  $m$  at the distance  $r = 3m$  (in the used system of units this Newtonian acceleration is equal to  $1/9m$ ).

It is interesting that the orbit  $r = 3m$  is a common solution of the MP Eqs. at the two known variants of the auxiliary conditions for these Eqs., namely, the condition of Pirani and Tulczyjew-Dixon [3]. Generally the solutions of the MP Eqs. at the different condition do not coincide.

It is clear that the solution of the MP Eqs. describing the orbit  $r = 3m$  in a Schwarzschild field is interesting mainly in the theoretical sense because in practice one cannot deal with a macroscopic particle moving with the ultrarelativistic velocity relatively the field source. There is much more perspective situation with the high-energy elementary particles, e.g. electrons or protons. In this connection the question arises: does the Dirac equation in a Schwarzschild field have a solution which corresponds, in the certain meaning, to the considered solution of the MP Eqs. with  $r = 3m$ ? For answer this question let us analyse the components of the 4-spinor  $\Psi_\mu$  which by the known procedure of separation of the variables in the Dirac equation in a Schwarzschild field (see, e.g., [10], Ch. 10) take the form

$$\begin{aligned} \Psi_1 &= \frac{1}{r\sqrt{2}}R_{-1/2}(r)S_{-1/2}(\theta)\exp[i(\sigma t + m'\varphi)], \\ \Psi_2 &= R_{+1/2}(r)S_{+1/2}(\theta)\exp[i(\sigma t + m'\varphi)], \\ \Psi_3 &= -R_{+1/2}(r)S_{-1/2}(\theta)\exp[i(\sigma t + m'\varphi)], \\ \Psi_4 &= -\frac{1}{r\sqrt{2}}R_{-1/2}(r)S_{+1/2}(\theta)\exp[i(\sigma t + m'\varphi)]. \end{aligned} \quad (23)$$

For the radial functions  $R_{+1/2}(r)$ ,  $R_{-1/2}(r)$  we have the expressions

$$R_{+1/2}(r) = \frac{1}{\sqrt{r^2 - 2mr}} \psi_{+1/2}(r) \exp\left(-\frac{i}{2} \arctan \frac{Mr}{\lambda}\right),$$

$$R_{-1/2}(r) = \psi_{-1/2}(r) \exp\left(+\frac{i}{2} \arctan \frac{Mr}{\lambda}\right), \quad (24)$$

where  $\lambda$  is the parameter of separation of the variables depended on the orbital moment,  $\sigma$  is the value of energy, and the functions  $\psi_{+1/2}(r)$ ,  $\psi_{-1/2}(r)$  can be find from the two differential equations written in [10]. We shall consider these Eqs. for the case  $mM \gg 1$ , that is when the Schwarzschild source is, e.g., an ordinary (not microscopic) black hole. When performing concrete calculations one can take into account different values of  $\sigma$ ,  $\lambda$  and investigate the corresponding quantum states. Here we consider the case when  $\sigma$  and  $\lambda$  are equal to the values of the energy and moment of the classical electron following from the MP Eqs. for the above considered circular orbit with  $r = 3m$ . Then for the functions  $\psi_{+1/2}$ ,  $\psi_{-1/2}$  we obtain the equations

$$\frac{d\psi_{+1/2}}{dx} - iA \left(1 - \frac{2}{x}\right)^{-1} \psi_{+1/2} + 3^{3/2} \frac{A}{x} \left(1 - \frac{2}{x}\right)^{-1/2} \psi_{-1/2} = 0,$$

$$\frac{d\psi_{-1/2}}{dx} + iA \left(1 - \frac{2}{x}\right)^{-1} \psi_{-1/2} + 3^{3/2} \frac{A}{x} \left(1 - \frac{2}{x}\right)^{-1/2} \psi_{+1/2} = 0, \quad (25)$$

where  $x \equiv r/m$ ,  $A \equiv 2m^3 M^3 / \sqrt{3}$  (Eqs. (25) are written for the values of  $x$  which are not in the small neighborhood of  $x = 2$ ). The analysis of the solutions of (25) shows that the property  $|\psi_{+1/2}| = |\psi_{-1/2}|$  takes place and the maximum value of  $|\psi_{\pm 1/2}|$  is achieved at  $x = 3$ . We stress that just the values  $|\psi_{\pm 1/2}|^2$  together with (23), (24) determine the probability to find an electron in the certain space region because for the components of the Dirac current  $J^\mu$  the relationship takes place [10]:

$$J^\mu = \sqrt{2} \left[ l^\mu (|\Psi_1|^2 + |\Psi_4|^2) + n^\mu (|\Psi_2|^2 + |\Psi_3|^2) - m^\mu (\Psi_1 \Psi_2^* - \Psi_3 \Psi_4^*) - m^{*\mu} (\Psi_1^* \Psi_2 - \Psi_3^* \Psi_4) \right], \quad (26)$$

where  $l^\mu, n^\mu, m^\mu, m^{*\mu}$  are the known isotropic vectors in the Newman-Penrose formalism. Taking into account (23), (26) and the relation  $|\psi_{+1/2}| = |\psi_{-1/2}|$  it is easy to find that  $J^\varphi \neq 0$ ,  $J^t \neq 0$ , whereas  $J^r = 0$ ,  $J^\theta = 0$ . It follows that the current circulates exactly on the circle and the maximum values of  $|J^\varphi|$ ,

$|J^t|$  are achieved at  $r = 3m$ . The width of the peak of the curve  $|\psi_{\pm 1/2}|^2$  decreases when  $A$  grows, and at  $A \rightarrow \infty$  we have the classical circular orbit with  $r = 3m$ .

So, the considered solution of the Dirac equation in a Schwarzschild field describes the quantum state corresponding to the classical orbit with  $r = 3m$ . We point out that the parameters  $\sigma$  and  $\lambda$  of this state are equal to the energy and moment of the classical electron on this orbit. As we stress above the circular orbit with  $r = 3m$  is an example when the gravitational ultrarelativistic spin-orbit acceleration becomes significant. Further on it is interesting to investigate other solutions of the Dirac equation which can show the role of the ultrarelativistic gravitation in the astrophysical processes.

## 6 Conclusions

Relationships (16), (17), and the conclusions drawn from them in concrete physical situations described by expressions (13), (19), (20), point unequivocally to the need for investigators to direct more attention to the gravitational interaction at ultrarelativistic relative velocities. In view of the correspondence principle, there are reasons to infer that some important relationships of gravitational interaction of classical (non-quantum) objects will, to a certain extent, hold for particles of the micro-world, where ultrarelativistic relative velocities and high energies are ubiquitous.

An important question asks whether the analyses, carried out above, might not be helpful in delving into the specifics of inclusion of the gravitational interaction into the scheme of unification of interactions. We think they might be. If only because a purely classical examination affords a deeper insight into the gravitational interaction in the micro-world at high energies.

Not less important is the need to elucidate how the entity which in classical terms is denoted as "gravitational ultrarelativistic spin-orbit interaction" should be expressed in the scheme of second quantization.



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