2022

The Topological Slice Genus Of Satellite Knots

P. Feller

Allison N. Miller
Swarthmore College, amille11@swarthmore.edu

J. Pinzón-Caicedo

Recommended Citation

P. Feller, Allison N. Miller, and J. Pinzón-Caicedo. (2022). "The Topological Slice Genus Of Satellite Knots". *Algebraic And Geometric Topology*. Volume 22, Issue 2. 709-738. DOI: 10.2140/agt.2022.22.709
https://works.swarthmore.edu/fac-math-stat/296

A NOTE ON THE CONCORDANCE $\mathbb{Z}$-GENUS

ALLISON N. MILLER AND JUNGHWAN PARK

ABSTRACT. We show that the difference between the topological 4-genus of a knot and the minimal genus of a surface bounded by that knot that can be decomposed into a smooth concordance followed by an algebraically simple locally flat surface can be arbitrarily large. This extends work of Hedden-Livingston-Ruberman showing that there are topologically slice knots which are not smoothly concordant to any knot with trivial Alexander polynomial.

1. INTRODUCTION

The $\mathbb{Z}$-*genus* of a knot K in S^3, denoted by $g_{\mathbb{Z}}(K)$, is the minimal genus of an oriented properly embedded locally flat surface Σ in the 4-ball B^4 such that the boundary of Σ is K and $\pi_1(B^4 \setminus \Sigma) \cong \mathbb{Z}$. Work of Freedman famously implies that a knot K has $g_{\mathbb{Z}}(K) = 0$ if and only if K has trivial Alexander polynomial [Fre82, FQ90, GT04]; later work of Feller generalizes Freedman's result to show that $g_{\mathbb{Z}}(K)$ is bounded above by half the degree of the Alexander polynomial of K [Fel16]. The $\mathbb{Z}$-genus of K can therefore be thought of as an algebraically controlled upper bound on the topological 4-ball genus of K, i.e. the minimal genus of *any* oriented properly embedded locally flat surface in the 4-ball with boundary K ; see Feller-Lewark for a precise statement of this fact [FL18, FL19]. Further, we define the (*smooth*) *concordance* $\mathbb{Z}$-*genus* of a knot K, denoted by $g^{c}_{\mathbb{Z}}(K)$, to be the minimum value of $g_{\mathbb{Z}}(J)$ among all knots smoothly concordant to J. That is,

$$g^{c}_{\mathbb{Z}}(K) := \min\{g_{\mathbb{Z}}(J) \mid J \text{ is smoothly concordant to } K\}.$$

Observe that a knot K is smoothly concordant to a knot with trivial Alexander polynomial if and only if $g^{c}_{\mathbb{Z}}(K) = 0$, and that we obtain the following inequalities for any knot K immediately from the definitions:

$$g_4^{\text{top}}(K) \leq g^{c}_{\mathbb{Z}}(K) \leq g_{\mathbb{Z}}(K),$$

where $g_4^{\text{top}}(K)$ denotes the topological 4-ball genus of K.

We remark that taking connected sums of a smoothly slice knot with nontrivial determinant produces knots with vanishing $g^{c}_{\mathbb{Z}}$ and arbitrarily large $g_{\mathbb{Z}}$ (see e.g. Proposition 2.1). In particular, this implies that the gap between $g^{c}_{\mathbb{Z}}$ and $g_{\mathbb{Z}}$ can be made arbitrarily large. In [HLR12], Hedden-Livingston-Ruberman used the Heegaard Floer correction terms of 3-manifolds to show that there is a topologically slice knot K which is not smoothly concordant to any knot with trivial Alexander polynomial, which in particular implies that $g_4^{\text{top}}(K) < g^{c}_{\mathbb{Z}}(K)$. We further show that the gap between these two invariants can be arbitrarily large.

Theorem 1.1. *There exist topologically slice knots with arbitrarily large concordance $\mathbb{Z}$-genus. More precisely, there exists a topologically slice knot K_* such that for each $n \in \mathbb{N}$ its n-fold connected self-sum has concordance $\mathbb{Z}$-genus at least n. Furthermore, we can choose K_* so that it has smooth 4-ball genus one.*

We outline the proof of the main result. We first construct a topologically slice knot K_* with smooth 4-ball genus one in Example 2.6. Let n be a positive integer and $\Sigma_2(\#^n K_*)$ be the 2-fold branched cover of S^3 branched over $\#^n K_*$. We then observe that for any nontrivial 5-torsion element x of $H_1(\Sigma_2(\#^n K_*))$, we may multiply an appropriate constant to obtain $z = c \cdot x$ so that the Heegaard Floer $\overline{d}$-invariant $\overline{d}(\Sigma_2(\#^n K_*), \mathfrak{s}_z)$ is nonvanishing. We use this to show that there is no rational homology cobordism from $\Sigma_2(\#^n K_*)$ to any 3-manifold Y with $H_1(Y)$ generated by at most $2n$ elements, and conclude that $g^{c}_{\mathbb{Z}}(\#^n K_*) \geq n$.

2020 *Mathematics Subject Classification.* 57K10, 57K40, 57K18, and 57N70.
The first author was partially supported by NSF grant DMS-1902880.

On a similar note, Hom [Hom15] showed that there are topologically slice knots with smooth 4-ball genus equal to one and arbitrarily large concordance genus. Recall that the concordance genus is greater than or equal to concordance $\mathbb{Z}$-genus since the fundamental group of the complement of a pushed in Seifert surface is equal to $\mathbb{Z}$.

Theorem 1.1 is related to the following difficult problem. We can analogously define the *topological concordance $\mathbb{Z}$-genus* of a knot K as

$$g_{\mathbb{Z}}^{\text{t.c.}}(K) := \min\{g_{\mathbb{Z}}(J) \mid J \text{ is topologically concordant to } K\}.$$

So for any knot K we have that

$$g_4^{\text{top}}(K) \leq g_{\mathbb{Z}}^{\text{t.c.}}(K) \leq g_{\mathbb{Z}}^{\text{c}}(K) \leq g_{\mathbb{Z}}(K).$$

Livingston [Liv04] showed that the gap between g_4^{top} and $g_{\mathbb{Z}}^{\text{t.c.}}$ can also be made arbitrarily large by using Casson-Gordon invariants (see Remark 2.11), but the following remains open.

Problem 1.2. Show that topological 4-ball genus can be strictly less than topological concordance smooth 4-ball genus, i.e. find a knot K with $g_4^{\text{top}}(K) = g \geq 1$ such that K is not topologically concordant to any knot J with $g_4(J) = g$, where $g_4(J)$ denotes the smooth 4-ball genus of J.

The following problem about properly embedded surfaces is a priori easier, but is nevertheless still open.

Problem 1.3. Find an oriented properly embedded locally flat genus $g \geq 1$ surface Σ in the 4-ball such that Σ is not ambiently isotopic rel boundary to any surface Σ' that can be decomposed into a topological concordance glued to a smoothly embedded surface.

Note that a positive solution to Problem 1.2 certainly implies a positive solution to Problem 1.3. While we avoid the technical details of topological 4-manifold theory needed to prove this formally, it is known both that every topologically slice knot is topologically concordant to the unknot, which of course has smooth 4-ball genus equal to zero, and that every topologically embedded disk can be decomposed into a topological concordance glued to the standard slice disk for the unknot. That is, the $g = 0$ case of Problems 1.2 and 1.3 is resolved.

Acknowledgments. This paper began in conversations during the first author's visit to the School of Mathematics at Georgia Institute for Technology to speak in the Geometry Topology Seminar; she thanks Miriam Kuzbary and the School as a whole for the invitation and for their generous hospitality. We also thank Peter Feller, Mark Powell, and Charles Livingston for helpful conversations.

Notation and conventions. In this paper, we work in the smooth category unless specified otherwise. Let $-K$ be the reverse of the mirror image of a knot K and $-Y$ be the manifold Y with reversed orientation. For any given knot K in S^3, the 2-fold branched cover of S^3 branched over K is denoted by $\Sigma_2(K)$, and for $n \in \mathbb{Z}$, the 3-manifold obtained by n-framed Dehn surgery on S^3 along K is denoted by $S^3_n(K)$.

2. Proof of Theorem 1.1

Feller-Lewark defined the *algebraic genus* of a knot K in [FL18] and further showed that this invariant coincides with the $\mathbb{Z}$-genus [FL19]. This implies the following key ingredient for the proof of Theorem 1.1. For a finite abelian group G, we denote by $r(G)$ the so-called "generating rank", i.e. the minimal number of generators of G.

Proposition 2.1 (Proposition 12 of [FL18])**.** *If $\Sigma_2(K)$ is the 2-fold branched cover of S^3 branched over a knot K, then*

$$r(H_1(\Sigma_2(K))) \leq 2g_{\mathbb{Z}}(K). \qquad \square$$

The other ingredient for the proof comes from an analysis of metabolizers for torsion linking forms. Recall that there is a nonsingular symmetric linking form

$$\lambda\colon H_1(Y) \times H_1(Y) \to \mathbb{Q}/\mathbb{Z},$$

for a rational homology sphere Y. If $-Y$ is the 3-manifold obtained by reversing the orientation of Y, then the linking form on $H_1(-Y)$ is given by $-\lambda$. We say that a subgroup $M \subseteq H_1(Y)$ is a metabolizer if $M = M^\perp$ with respect to λ; note that for a metabolizer M the nonsingularity of λ immediately implies that $|M|^2 = |H_1(Y)|$. It is well known that if Y bounds a rational homology ball W, then the kernel of the inclusion-induced map $H_1(Y) \to H_1(W)$ is a metabolizer for $(H_1(Y), \lambda)$.

For a finite abelian group G and a prime p, let G_p denote the p-primary subgroup of G, i.e. the subgroup consisting of all elements of G which are p^k-torsion for some positive integer k.

Lemma 2.2. *Let Y_1 and Y_2 be rational homology 3-spheres and let p be a prime integer. Furthermore, for $i = 1, 2$, let λ_i be the linking form on $H_1(Y_i)$, and let M be a metabolizer for $(H_1(Y_1\# - Y_2), \lambda_1 \oplus -\lambda_2)$.*

If $H_1(Y_1)_p \cong (\mathbb{Z}/p^2\mathbb{Z})^{2n}$ and $r(H_1(Y_2)_p) \leq 2m < 2n$ for nonnegative integers n and m, then $M \cap H_1(Y_1)$ contains a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{n-m}$.

Proof. First, note that M splits as a direct sum of its q-primary components over all prime q, each of which is a metabolizer for the q-primary component of $H_1(Y_1\#-Y_2)$. We can therefore assume without loss of generality that $H_1(Y_1)$ and $H_1(-Y_2)$ are equal to their p-primary components, and do so for ease of notation.

For $i = 1, 2$, let $\pi_i \colon M \to H_1(Y_i)$ be the restriction of the projection map

$$H_1(Y_1\# - Y_2) \cong H_1(Y_1) \oplus H_1(-Y_2) \to H_1(Y_i).$$

Observe that $\operatorname{Ker}(\pi_2) = M \cap H_1(Y_1)$, which will be used later. We now argue that $p\operatorname{Im}(\pi_2)$ has the property that $-\lambda_2|_{p\operatorname{Im}(\pi_2)\times p\operatorname{Im}(\pi_2)} = 0$. If $b_1, b_2 \in \operatorname{Im}(\pi_2)$, then there exist $a_1, a_2 \in H_1(Y_1)$ such that $(a_1, b_1), (a_2, b_2) \in M$, and

$$(\lambda_1 \oplus -\lambda_2)((a_1, b_1), (a_2, b_2)) = \lambda_1(a_1, a_2) - \lambda_2(b_1, b_2) = 0 \in \mathbb{Q}/\mathbb{Z}.$$

Since $H_1(Y_1)$ is annihilated by p^2, we have that

$$-\lambda_2(pb_1, pb_2) = -p^2\lambda_2(b_1, b_2) = -p^2\lambda_1(a_1, a_2) = -\lambda_1(p^2a_1, a_2) = 0 \in \mathbb{Q}/\mathbb{Z},$$

as desired. Moreover, since the linking form $-\lambda_2$ is nonsingular, we have

$$|p\operatorname{Im}(\pi_2)| \leq |H_1(-Y_2)|^{1/2}. \tag{2.1}$$

Recall that since we are assuming $r(H_1(Y_2)) \leq 2m$ and since $\operatorname{Im}(\pi_2)$ is a subgroup of $H_1(-Y_2)$, we have that

$$\operatorname{Im}(\pi_2) \cong \bigoplus_{k=1}^{\ell}(\mathbb{Z}/p^k\mathbb{Z})^{n_k} \quad \text{and} \quad p\operatorname{Im}(\pi_2) \cong \bigoplus_{k=1}^{\ell}(\mathbb{Z}/p^{k-1}\mathbb{Z})^{n_k}, \quad \text{where} \quad \sum_{k=1}^{\ell} n_k \leq 2m.$$

Therefore,

$$\frac{|\operatorname{Im}(\pi_2)|}{|p\operatorname{Im}(\pi_2)|} = \frac{p^{\sum_{k=1}^{\ell} kn_k}}{p^{\sum_{k=1}^{\ell}(k-1)n_k}} = p^{\sum_{k=1}^{\ell} n_k} \leq p^{2m}.$$

So, by combining inequality (2.1) with the above inequality, we conclude that

$$|\operatorname{Im}(\pi_2)| \leq p^{2m}|H_1(-Y_2)|^{1/2}.$$

Also, recall that $|M| = |H_1(Y_1)|^{1/2}|H_1(-Y_2)|^{1/2}$ and $\operatorname{Ker}(\pi_2) = M \cap H_1(Y_1)$. Therefore,

$$|M \cap H_1(Y_1)| = |\operatorname{Ker}(\pi_2)| = \frac{|M|}{|\operatorname{Im}(\pi_2)|} \geq \frac{|H_1(Y_1)|^{1/2}|H_1(-Y_2)|^{1/2}}{p^{2m}|H_1(-Y_2)|^{1/2}} = p^{2n-2m}.$$

Finally, since $M \cap H_1(Y_1)$ is a subgroup of $(\mathbb{Z}/p^2\mathbb{Z})^{2n}$, we conclude that

$$M \cap H_1(Y_1) \cong (\mathbb{Z}/p\mathbb{Z})^{k_1} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{k_2}, \quad \text{where} \quad k_1 + 2k_2 \geq 2n - 2m.$$

Therefore,

$$2n - 2m \leq k_1 + 2k_2 \leq 2k_1 + 2k_2,$$

and we conclude that $M \cap H_1(Y_1)$ contains a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{n-m}$. □

Now, we collect some necessary background on the Heegaard-Floer correction term $d(Y, \mathfrak{s}) \in \mathbb{Q}$ associated to a rational homology sphere Y with a Spin^c structure $\mathfrak{s}$ [OS03b]. We first recall the following definition from [HLR12].

Definition 2.3 ([HLR12, Definition 2.2 and 3.1]). Let Y be a $\mathbb{Z}/2\mathbb{Z}$-homology sphere. For $z \in H_1(Y)$, let $\mathfrak{s}_z$ be the unique Spin^c structure of Y which satisfies $c_1(\mathfrak{s}_z) = 2\widehat{z} \in H^2(Y)$ where $\widehat{z}$ is the Poincaré dual of z. In particular, $\mathfrak{s}_0$ is the unique Spin structure on Y. Lastly, we define $\overline{d}(Y, \mathfrak{s}_z) := d(Y, \mathfrak{s}_z) - d(Y, \mathfrak{s}_0)$.

Further, the correction term is additive under connected sums [OS03a, Theorem 4.3] and is a Spin^c rational homology cobordism invariant [OS03a, Theorem 1.2]. Hence we get the following lemma (see e.g. [HLR12, Proposition 2.1]).

Lemma 2.4. *If K and J are two concordant knots, then there exists a metabolizer $M \subseteq H_1(\Sigma_2(K)\# - \Sigma_2(J))$ such that for each element $(m_K, m_J) \in M$ where $m_K \in H_1(\Sigma_2(K))$ and $m_J \in H_1(-\Sigma_2(J))$, we have that $d(\Sigma_2(K), \mathfrak{s}_{m_K}) + d(-\Sigma_2(J), \mathfrak{s}_{m_J}) = 0$.* □

We are now ready to prove the following corollary.

Corollary 2.5. *Let K be a knot with $g^c_{\mathbb{Z}}(K) = m$ and p be a prime integer. If $H_1(\Sigma_2(K))_p \cong (\mathbb{Z}/p^2\mathbb{Z})^{2n}$ and $n > m$, then there exists a subgroup $N \subseteq H_1(\Sigma_2(K))_p$ such that $N \cong (\mathbb{Z}/p\mathbb{Z})^{n-m}$ and $\overline{d}(\Sigma_2(K), \mathfrak{s}_z) = 0$ for each $z \in N$.*

Proof. Suppose K is concordant to a knot J with $g_{\mathbb{Z}}(J) = m$. Then by Proposition 2.1 $r(H_1(\Sigma_2(J))) \leq 2m$. As in the proof of Lemma 2.2 we assume that $H_1(\Sigma_2(K))$ and $H_1(-\Sigma_2(J))$ are equal to their p-primary components.

By Lemma 2.4, there exists a metabolizer $M \subseteq H_1(\Sigma_2(K)\# - \Sigma_2(J))$ such that for each element $(m_K, m_J) \in M$ where $m_K \in H_1(\Sigma_2(K))$ and $m_J \in H_1(-\Sigma_2(J))$,

$$d(\Sigma_2(K), \mathfrak{s}_{m_K}) + d(-\Sigma_2(J), \mathfrak{s}_{m_J}) = 0. \tag{2.2}$$

Lastly, since $\Sigma_2(K)$ and $\Sigma_2(J)$ satisfy the assumption of Lemma 2.2, we may conclude that $M \cap H_1(Y_1)$ contains a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{n-m}$. Set $N := M \cap H_1(Y_1)$, then note that for each $z \in N$, we have $(0, 0), (z, 0) \in N$. The proof is complete by using equation (2.2). □

Example 2.6. Our examples come from the cabling construction. Let $K_{p,q}$ denote the (p, q)-cable of K, where p is the longitudinal winding. We define our knot to be the following

$$K_* := D_{2,25}\# - D_{2,23}\# - T_{2,25}\#T_{2,23},$$

where D is the 4-fold connected self-sum of the positive Whitehead double of the right-handed trefoil knot. Since D is topologically slice, $D_{p,q}$ is topologically concordant to the torus knot $T_{p,q}$. Therefore K_* is topologically concordant to

$$K'_* := T_{2,25}\# - T_{2,23}\# - T_{2,25}\#T_{2,23},$$

hence is topologically slice. Also, note that by performing two crossing changes with opposite signs on K_* (one in the $D_{2,25}$ summand and one in the $-T_{2,25}$ summand), we obtain the smoothly slice knot $D_{2,23}\# - D_{2,23}\# - T_{2,23}\#T_{2,23}$. Hence K_* bounds a smoothly embedded genus one surface in the 4-ball. Finally, by [AK80], the 2-fold branched cover of S^3 branched over K_* is given by

$$\Sigma_2(K_*) \cong S^3_{25}(D\#D^r)\# - S^3_{23}(D\#D^r)\# - S^3_{25}(U)\#S^3_{23}(U),$$

where D^r is a knot obtained by reversing the orientation of D.

Next, we compute the correction terms for the 2-fold branched cover. Recall that for any knot K in S^3, there are nonnegative integer-valued smooth concordnace invariants $V_i(K)$, introduced by Rasmussen [Ras03]. Furthermore, Ni and Wu [NW15] showed that the correction terms of all Dehn surgeries on K can be computed from these invariants.

Proposition 2.7 ([NW15, Proposition 1.6 and Remark 2.10]). *Let K be a knot and U be the unknot. If n is a positive integer, then*

$$d(S^3_n(K), \mathfrak{s}_i) = d(S^3_n(U), \mathfrak{s}_i) - 2\max\{V_i(K), V_{n-i}(K)\}.$$ □

Furthermore, the correction terms for Dehn surgeries on the unknot are computed in [OS03a].

Proposition 2.8 ([OS03a, Proposition 4.8]). *If U is the unknot and n is a positive integer, then*

$$d(S^3_n(U), \mathfrak{s}_i) = \frac{(n-2i)^2}{4n} - \frac{1}{4}.$$ □

Hom and Wu [HW16] introduced a nonnegative integer valued smooth concordance invariant ν^+. Moreover, for any knot K, we have that $\nu^+(K) = 0$ if and only if $V_0(K) = 0$ [HW16, Proposition 2.3]. Following [KP18] (see also [Hom17]), we say that two knots K and K' are ν^+*-equivalent* if

$$\nu^+(K\# - K') = \nu^+(K'\# - K) = 0,$$

and it forms a equivalence relation on the set of concordance classes of knots. It is well-known that two ν^+-equivalent knots have the same V_i invariants (see e.g. [KKP19, Proposition 3.11]). We will use the fact that $D\#D^r$ is ν^+ equivalent to the torus knot $T_{2,17}$. This fact might be well-known to experts, but we sketch the proof here for the reader's convenience.

Proposition 2.9. *The knot $D\#D^r$ in Example 2.6 is ν^+-equivalent to the torus knot $T_{2,17}$. In particular, we have*

$$V_i(D\#D^r) = \begin{cases} 4 & \text{for } i = 0, \\ 2 & \text{for } i = 5, \\ 0 & \text{for } i \geq 8. \end{cases}$$

Proof. If K_i and K_i' are ν^+-equivalent for $i = 0, 1$, then $K_0\#K_1$ and $K_0'\#K_1'$ are also ν^+-equivalent (see e.g. [KKP19, Proposition 3.12]). Also, D^r is ν^+-equivalent to D since their knot Floer complexes are chain homotopy equivalent [OS04, Proposition 3.9]. Hence we have that $D\#D^r$ is ν^+-equivalent to the 8-fold connected self-sum of the positive Whitehead double of the right-handed trefoil knot. Lastly, we apply a result of Hedden, Kim, and Livingston [HKL16, Proposition 6.1] which states that the n-fold connected self-sum of the positive Whitehead double of the right-handed trefoil knot is ν^+-equivalent to $T_{2,2n+1}$. This proves the first part of the statement.

The last part follows from two facts. First, as mentioned above, two ν^+-equivalent knots have the same V_i invariants. The second fact is that V_i invariants for any alternating knot K are determined by their signatures as follows (see e.g. [HW16, Theorem 2] and [AG17, Section 5.1]):

$$V_i(K) = \max\left\{\left\lceil -\frac{\sigma(K) + 2i}{4}\right\rceil, 0\right\}.$$

This completes the proof. ◻

Now, we are ready to compute the correction terms.

Proposition 2.10. *Let K_* be the knot in Example 2.6 and $\#^n K_*$ be the n-fold connected self-sum of K_*. If $N \subseteq H_1(\Sigma_2(\#^n K_*))_5$ is a nontrivial subgroup, then there exists an element $z \in N$ such that $\overline{d}(\Sigma_2(\#^n K_*), \mathfrak{s}_z) \neq 0$.*

Proof. By Example 2.6, we have $\Sigma_2(K_*) \cong S^3_{25}(D\#D^r)\# - S^3_{23}(D\#D^r)\# - S^3_{25}(U)\#S^3_{23}(U)$ and $H_1(\Sigma_2(K_*))_5 \cong \mathbb{Z}/25\mathbb{Z} \oplus \mathbb{Z}/25\mathbb{Z} \cong H_1(S^3_{25}(D\#D^r)\# - S^3_{25}(U))$. Note that by the additivity of the correction terms, we have that

$$\overline{d}(\Sigma_2(K_*), \mathfrak{s}_{(i,j)}) = \overline{d}(\Sigma_2(K_*), \mathfrak{s}_{(i,0)}) + \overline{d}(\Sigma_2(K_*), \mathfrak{s}_{(0,j)}).$$

For $(i, 0) \in H_1(\Sigma_2(K_*))_5$, we have by Proposition 2.7, Proposition 2.8, and Proposition 2.9 that

$$\begin{aligned} \overline{d}(\Sigma_2(K_*), \mathfrak{s}_{(i,0)}) &= \overline{d}(S^3_{25}(U), \mathfrak{s}_i) - 2\max\{V_i(D\#D^r), V_{25-i}(D\#D^r)\} + 2V_0(D\#D^r) \\ &= \begin{cases} 0 & \text{for } i = 5 \text{ and } 20, \\ 2 & \text{for } i = 10 \text{ and } 15. \end{cases} \end{aligned}$$

Similarly, for $(0, j) \in H_1(\Sigma_2(K_*))_5$ we have that

$$\overline{d}(\Sigma_2(K_*), \mathfrak{s}_{(0,j)}) = -\overline{d}(S^3_{25}(U), \mathfrak{s}_j) = \begin{cases} 4 & \text{for } j = 5 \text{ and } 20, \\ 6 & \text{for } j = 10 \text{ and } 15. \end{cases}$$

By additivity, we have that $\overline{d}(\Sigma_2(K_*), \mathfrak{s}_x) \geq 0$ for any order 5 element $x \in H_1(\Sigma_2(K_*))_5$. Moreover, at least one of $\overline{d}(\Sigma_2(K_*), \mathfrak{s}_x)$ and $\overline{d}(\Sigma_2(K_*), \mathfrak{s}_{2x})$ must be strictly positive.

Now, let $N \subseteq H_1(\Sigma_2(\#^n K_*))_5$ be a nontrivial subgroup, and let y be an order 5 element in N. Note that since $\Sigma_2(\#^n K_*) \cong \#^n \Sigma_2(K_*)$, we can naturally write $y = (y_1, \ldots, y_n)$ for $y_i \in H_1(\Sigma_2(K_*))_5$ and have that

$$\overline{d}(\Sigma_2(\#^n K_*), \mathfrak{s}_y) = \sum_{i=1}^{n} \overline{d}(\Sigma_2(K_*), \mathfrak{s}_{y_i}).$$

Therefore, at least one of $\overline{d}(\Sigma_2(\#^n K_*), \mathfrak{s}_y)$ and $\overline{d}(\Sigma_2(\#^n K_*), \mathfrak{s}_{2y})$ is strictly positive. □

We are now ready to prove our main theorem. We recall the statement.

Theorem 1.1. *There exist topologically slice knots with arbitrarily large concordance $\mathbb{Z}$-genus. More precisely, there exists a topologically slice knot K_* such that for each $n \in \mathbb{N}$ its n-fold connected self-sum has concordance $\mathbb{Z}$-genus at least n. Furthermore, we can choose K_* so that it has smooth 4-ball genus one.*

Proof. Let K_* be the knot in Example 2.6. It is topologically slice and has smooth 4-ball genus at most one as we observed above.

We will show that $\#^n K_*$, the n-fold connected self-sum of K_*, has concordance $\mathbb{Z}$-genus at least n. Suppose $g_{\mathbb{Z}}^{\mathrm{c}}(\#^n K_*) = m$ and $n > m$, then by Corollary 2.5 there exists a nontrivial subgroup $N \subseteq H_1(\Sigma_2(\#^n K_*))_5$ such that $\overline{d}(\Sigma_2(\#^n K_*), \mathfrak{s}_z) = 0$ for each $z \in N$. This contradicts Proposition 2.10. □

Remark 2.11. A similar though much more involved argument using Casson-Gordon signatures instead of d-invariants can be used to show that the difference between the topological 4-ball genus and the topological concordance $\mathbb{Z}$-genus can be arbitrarily large.

In fact, a careful reading of Livingston's paper [Liv04] shows that, despite the fact that it only explicitly considers the topological concordance genus (i.e. the minimal Seifert genus of any representative of a given topological concordance class), all the relevant work has been done to prove that for each $n \in \mathbb{N}$ there exists a knot K_n with the following properties:

(1) K_n has topological 4-ball genus one.
(2) K_n is not topologically concordant to any knot J with $r(H_1(\Sigma_2(J))) < 2n$.

In particular, by Proposition 2.1, this implies that K_n is not topologically concordant to any knot J with $g_{\mathbb{Z}}(J) < n$.

This result of Livingston is relevant to understanding how g_4^{top} changes under the satellite operation. Classical techniques show that for each pattern P there is a constant $g_P \geq g_4(P(U))$ such that for any knot K we have

$$g_4(P(K)) \leq |w_P| \cdot g_4(K) + g_P,$$

where w_P is the algebraic winding number of the pattern P, and modern Heegaard Floer invariants can be used to show that this inequality is sometimes sharp [FMPC19].

We similarly see that for each P there exists a constant $g_P^{\mathrm{top}} \geq g_4^{\mathrm{top}}(P(U))$ such that for any knot K we have

$$g_4^{\mathrm{top}}(P(K)) \leq |w_P| \cdot g_4^{\mathrm{top}}(K) + g_P^{\mathrm{top}}, \tag{2.3}$$

but now the sharpness of this result is far from clear–in fact, it remains possible that the factor coming from winding number is unnecessary. More precisely, it is not known if the following inequality holds for every knot K:

$$g_4^{\mathrm{top}}(P(K)) \leq g_4^{\mathrm{top}}(K) + g_4^{\mathrm{top}}(P(U)). \tag{2.4}$$

Work of Feller, Miller, and Pinzon-Caicedo [FMPC19] and independently McCoy [McC19] shows that for P a pattern with $P(U) = U$ one has

$$g_4^{\mathrm{top}}(P(K)) \leq g_{\mathbb{Z}}^{\mathrm{t.c.}}(K), \tag{2.5}$$

Hence, Livingston's examples of knots K with $g_4^{\mathrm{top}}(K) < g_{\mathbb{Z}}^{\mathrm{t.c.}}(K)$ are good candidates for input companion knots if one wishes to show that the inequality (2.4) is not true.

More concretely, we state the following problem.

Problem 2.12. Let K_n be Livingston's nth knot, which has $g_4^{\text{top}}(K) = 1$ and $g_{\mathbb{Z}}^{\text{t.c.}}(K) \geq n$. If the inequality (2.4) is true, then for any positive integers n, m we have that

$$g_4^{\text{top}}(C_{m,1}(K_n)) = 1.$$

However, the best upper bounds we have coming from the topological concordance $\mathbb{Z}$-genus (i.e. the inequality (2.5)) and from classical arguments (i.e. the inequality (2.3)) are

$$g_4^{\text{top}}(C_{m,1}(K_n)) \leq \min\{m, g_{\mathbb{Z}}^{\text{t.c.}}(K_n)\}.$$

Determine $g_4^{\text{top}}(C_{m,1}(K_n))$ for some $m, n > 1$.

References

[AG17] Paolo Aceto and Marco Golla. Dehn surgeries and rational homology balls. *Algebr. Geom. Topol.*, 17(1):487–527, 2017.

[AK80] Selman Akbulut and Robion Kirby. Branched covers of surfaces in 4-manifolds. *Math. Ann.*, 252(2):111–131, 1979/80.

[Fel16] Peter Feller. The degree of the Alexander polynomial is an upper bound for the topological slice genus. *Geom. Topol.*, 20(3):1763–1771, 2016.

[FL18] Peter Feller and Lukas Lewark. On classical upper bounds for slice genera. *Selecta Math. (N.S.)*, 24(5):4885–4916, 2018.

[FL19] Peter Feller and Lukas Lewark. Balanced algebraic unknotting, linking forms, and surfaces in three- and four-space. arXiv:1905.08305, 2019.

[FMPC19] Peter Feller, Allison N. Miller, and Juanita Pinzon-Caicedo. A note on the topological slice genus of satellite knots. arXiv:1908.03760, to appear in Algebr. Geom. Topol., 2019.

[FQ90] Michael H. Freedman and Frank Quinn. *Topology of 4-manifolds*, volume 39 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1990.

[Fre82] Michael H. Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.

[GT04] Stavros Garoufalidis and Peter Teichner. On knots with trivial Alexander polynomial. *J. Differential Geom.*, 67(1):167–193, 2004.

[HKL16] Matthew Hedden, Se-Goo Kim, and Charles Livingston. Topologically slice knots of smooth concordance order two. *J. Differential Geom.*, 102(3):353–393, 2016.

[HLR12] Matthew Hedden, Charles Livingston, and Daniel Ruberman. Topologically slice knots with nontrivial Alexander polynomial. *Adv. Math.*, 231(2):913–939, 2012.

[Hom15] Jennifer Hom. On the concordance genus of topologically slice knots. *Int. Math. Res. Not. IMRN*, (5):1295–1314, 2015.

[Hom17] Jennifer Hom. A survey on Heegaard Floer homology and concordance. *J. Knot Theory Ramifications*, 26(2):1740015, 24, 2017.

[HW16] Jennifer Hom and Zhongtao Wu. Four-ball genus bounds and a refinement of the Ozváth-Szabó tau invariant. *J. Symplectic Geom.*, 14(1):305–323, 2016.

[KKP19] Min Hoon Kim, David Krcatovich, and JungHwan Park. Links with non-trivial Alexander polynomial which are topologically concordant to the Hopf link. *Trans. Amer. Math. Soc.*, 371(8):5379–5400, 2019.

[KP18] Min Hoon Kim and Kyungbae Park. An infinite-rank summand of knots with trivial Alexander polynomial. *J. Symplectic Geom.*, 16(6):1749–1771, 2018.

[Liv04] Charles Livingston. The concordance genus of knots. *Algebr. Geom. Topol.*, 4:1–22, 2004.

[McC19] Duncan McCoy. Null-homologous twisting and the algebraic genus. arXiv:1908.04043, to appear in 2019 MATRIX Annals, 2019.

[NW15] Yi Ni and Zhongtao Wu. Cosmetic surgeries on knots in S^3. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2015(706):1–17, 2015.

[OS03a] Peter Ozsváth and Zoltán Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Advances in Mathematics*, 173(2):179–261, 2003.

[OS03b] Peter Ozsváth and Zoltán Szabó. Heegaard Floer homology and alternating knots. *Geom. Topol.*, 7:225–254, 2003.

[OS04] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and knot invariants. *Advances in Mathematics*, 186(1):58–116, 2004.

[Ras03] Jacob A. Rasmussen. *Floer homology and knot complements*. ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–Harvard University.

Department of Mathematics, Rice University, Houston, TX, United States
Email address: allison.miller@rice.edu

Department of Mathematical Sciences, KAIST, Daejeon, South Korea
Email address: jungpark0817@kaist.ac.kr