4-1-2022

Amphichiral Knots With Large 4-Genus

Allison N. Miller
Swarthmore College, amille11@swarthmore.edu

Recommended Citation
Allison N. Miller. (2022). "Amphichiral Knots With Large 4-Genus". *Bulletin Of The London Mathematical Society.* Volume 54, Issue 2. 624-634. DOI: 10.1112/blms.12588
https://works.swarthmore.edu/fac-math-stat/295

AMPHICHIRAL KNOTS WITH LARGE 4-GENUS

ALLISON N. MILLER

ABSTRACT. For each $g > 0$ we give infinitely many knots that are strongly negative amphichiral, hence rationally slice and representing 2-torsion in the smooth concordance group, yet which do not bound any locally flatly embedded surface in the 4-ball with genus less than or equal to g. Our examples also allow us to answer a question about the 4-dimensional clasp number of knots.

1. Introduction

A knot K in S^3 is called *strongly negative amphichiral* if there exists an orientation reversing involution $\varphi\colon S^3 \to S^3$ such that $\varphi(K) = K$. Many concordance invariants vanish on such knots, including the classical Tristram-Levine signature function [Lev69, Tri69] and more modern invariants coming from Heegaard Floer and Khovanov homology like the τ-invariant [OS03], ν^+-invariant [HW16], Υ-invariant [OSS17], s-invariant [Ras10], s_n-invariants [Lob09, Wu09], $s^\#$-invariant [KM13], and $\daleth$-invariant [LL19]. Notably, this list contains almost all known lower bounds on the *4-genus*, or minimal genus of a (smoothly or locally flatly) embedded orientable surface in B^4 with boundary the given knot. However, we use Gilmer's bound on the topological 4-genus [Gil82] coming from Casson-Gordon signatures [CG86] to prove the following.

Theorem 1.1. *For any $g > 0$, there exists a knot K with the following properties:*

(1) K is strongly negative amphichiral.
(2) K can be transformed to a smoothly slice knot by either (a) changing some crossings $(+)$ to $(-)$ or (b) changing some crossings $(-)$ to $(+)$.
(3) the topological 4-genus of K is strictly larger than g.

In fact, something more is true, and proven in Proposition 2.7: for any $g \in \mathbb{N}$ there exists an infinite family of knots $\{K^k\}_{k\in\mathbb{N}}$, generating a subgroup of the concordance group isomorphic to $(\mathbb{Z}_2)^\infty$, such that any nontrivial sum $K = \#_{j=1}^m K^{k_j}$ satisfies the conclusions of Theorem 1.1. Moreover, each of the knots K^k is algebraically slice, so we incidentally reprove a result of Livingston [Liv99] that there is a $(\mathbb{Z}_2)^\infty$-subgroup of the concordance group consisting of algebraically slice knots.

Negative amphichiral knots, if not slice, represent 2-torsion elements of the smooth concordance group; a still-open question of Gordon asks whether all 2-torsion elements have such representatives [Hau78, Problem 16]. We therefore obtain the following corollary to Theorem 1.1, which appears to be previously unknown.

Corollary 1.2. *There exist 2-torsion knots with arbitrarily large 4-genera.*

A knot K is called *rationally slice* if there exists a smooth 4-manifold W with boundary $\partial W = S^3$ and $H_*(W;\mathbb{Q}) = H_*(B^4;\mathbb{Q})$ such that K bounds a smoothly embedded null-homologous disc in W. Every strongly negative amphichiral knot is rationally slice [Kaw09], and so Theorem 1.1 also answers a question of [HKJPS20] in the affirmative.

Corollary 1.3. *There exist rationally slice knots with arbitrarily large 4-genera.*

2020 *Mathematics Subject Classification.* 57K10 .

Key words and phrases. Casson-Gordon signatures, clasp number, four genus, rationally slice, strongly negative amphichiral.

The *4-dimensional clasp number* $c_4(K)$ of a knot K is the minimal number of transverse double points across all immersions of D^2 in B^4 with $\partial D^2 = K$. Similarly, $c_4^+(K)$ (respectively $c_4^-(K)$) is defined to be the minimal number of positive (resp. negative) transverse double points across all immersions of D^2 in B^4 with $\partial D^2 = K$. It follows immediately from the definitions that $c_4^+ + c_4^- \leq c_4$; the figure-eight knot 4_1 is the prototypical example of when this inequality is strict, since $c_4^+(4_1) = c_4^-(4_1) = 0$ and yet $c_4(4_1) = 1$. We answer a question of [JZ20] by giving the first examples of knots for which $c_4(K)$ is arbitrarily larger than $c_4^+(K) + c_4^-(K)$.

Corollary 1.4. *The difference between $c_4(K)$ and $c_4^+(K) + c_4^-(K)$ can be arbitrarily large.*

Proof. For $g \in \mathbb{N}$, let K_g be a knot satisfying the conclusions of Theorem 1.1. By (2), we have that $c_4^+(K_g) + c_4^-(K_g) = 0 + 0 = 0$, and by (3) we have that

$$g < g_4(K_g) \leq g_4^s(K_g) \leq c_4(K_g),$$

noting that standard arguments show that for any knot K the smooth 4-genus $g_4^s(K)$ is bounded above by $c_4(K)$. □

Since Casson-Gordon signatures provide bounds on the topological 4-genus, it remains open whether one can find examples for the smooth analogue of Theorem 1.1 as follows.

Question 1.5. For $g \in \mathbb{N}$, does there exist a topologically slice knot K such that $g_4^s(K) > g$ and

(1) K is order 2 in the smooth concordance group?
(2) K is smoothly rationally slice?
(3) $c_4^+(K) = c_4^-(K) = 0$?

Recent work of Hom-Kang-Park-Stoffregen [HKJPS20] has shown that $\{C_{2n+1,1}(4_1)\}_{n\in\mathbb{N}}$ generates a $\mathbb{Z}^\infty$-subgroup of rationally slice knots in the smooth concordance group. By work of [FMPC19], the topological 4-genus of $C_{2n+1,1}(4_1)$ equals 1 for all $n \in \mathbb{N}$, but it remains open whether the smooth 4-genus of $C_{2n+1,1}(4_1)$ is large. Since $2n+1$ is relatively prime to 2, one can combine the work of this paper with the formulas for Casson-Gordon signatures of satellite knots given in [Lit84] and conclude that for our choice of K_g satisfying the conclusions of Theorem 1.1, we have that $g_4(C_{2n+1,1}(K_g)) > g$ for all $n \in \mathbb{N}$. We therefore state the following as an interesting open problem in either the smooth or topological categories.

Question 1.6. For any $g \in \mathbb{N}$, let K_g be one of the knots given in Section 2 that satisfies the conclusions of Theorem 1.1. For some or all $n \in \mathbb{N}$, determine whether $C_{2n+1,1}(K_g)$ is infinite order in the concordance group.

We note that it remains open even whether $C_{2n,1}(K)$ must always be slice for strongly negative amphichiral K, though it is known that many such knots are not ribbon [Miy94].

Remark 1.7. The key feature of Casson-Gordon signatures that allows us to use Gilmer's bound to establish Theorem 1.1 when all other lower bounds on the 4-genus fail might initially seem like a flaw: no single signature gives a 4-genus bound or even a sliceness obstruction. While we avoid stating the precise definition of these invariants, we remind the reader that $\sigma(K,\chi) \in \mathbb{Q}$ depends on not just the knot K but a choice of map χ from the first homology of the double branched cover of K to a cyclic group. The fact that K is negative amphichiral implies that there is an involution ι on the set of such maps such that $\sigma(K,\iota(\chi)) = -\sigma(K,\chi)$. As long as this involution is non-trivial, the negative amphichirality of K does not force $\sigma(K,\chi)$ to vanish and there is still the potential to obtain a sliceness obstruction–and even a lower bound on the 4-genus–by considering the set of all such signatures. This could be considered as philosophically similar to the fact that Casson-Gordon signatures can obstruct knots from being concordant to their reverses [KL99], though that result requires a careful analysis of additional structure that we are able to avoid.

Acknowledgments

The author is indebted to Anthony Conway and JungHwan Park for thoughtful conversations and for suggesting the questions resolved by Corollaries 1.2 and 1.4, and to Chuck Livingston for asking about algebraic sliceness. The author also gratefully acknowledges her partial support by NSF DMS-1902880.

2. Proof of Main Result

Our examples are connected sums of certain satellites of the figure-eight knot.

Example 2.1. Let J be a reversible knot and define $K(J)$ to be as in Figure 1, where $\overline{J}$ denotes the mirror image of J, which since J is reversible equals the concordance inverse $-J$. The right

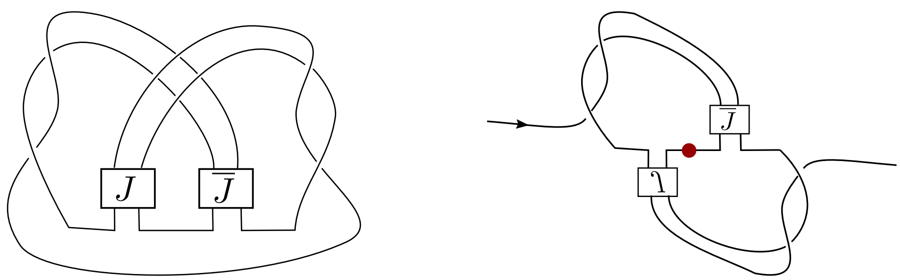

Figure 1. The knot $K(J)$ from 2 perspectives.

side of Figure 1 demonstrates that $K(J)$ is strongly negative amphichiral: rotation by 180 degrees in the plane about the marked point followed by reflection in the plane of the page takes $K(J)$ to itself. Also observe that the disc-with-bands Seifert surface for $K(J)$ visible on the left of Figure 1 demonstrates that $K(J)$ shares a Seifert form with the figure-eight knot K_0.

Proposition 2.2. *If J is a reversible knot, then $K(J)$ has $c_4^+(K_J) = c_4^-(K_J) = 0$.*

Proof. Consider the knots $K_\pm$ as depicted in Figure 2, shown with genus one Seifert surfaces $F_\pm$ in disc-with-bands position. Observe that K_+ (respectively K_-) is obtained from K_J by changing a single negative (resp. positive) crossing to a positive (resp. negative) crossing. Figure 2 also

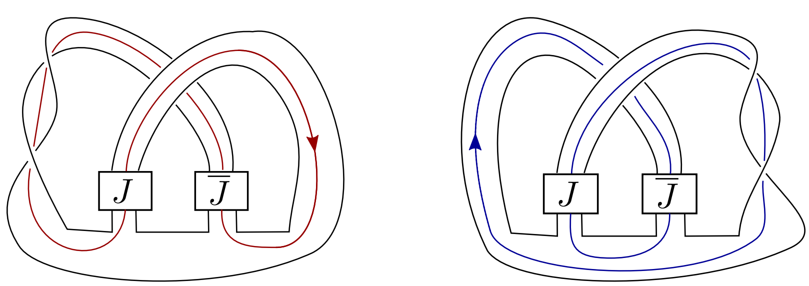

Figure 2. K_+, obtained by changing a crossing from $-$ to $+$ (left) and K_-, obtained by changing a crossing from $+$ to $-$ (right).

depicts a curve $\gamma_\pm$ on $F_\pm$. Note that each of $\gamma_\pm$ represents a nontrivial element of $H_1(F_\pm)$ and is 0-framed by $F_\pm$; i.e. is an *derivative curve*. Considered as a knot, γ_+ is $J\#\overline{J}$; since J is reversible this is isotopic to $J\#-J$ and hence is slice. Similarly, the knot type of γ_- is the slice knot $J\#-J$. Therefore, surgering the Seifert surface $F_\pm$ along the derivative curve $\gamma_\pm$ yields a smooth slice disc for $K_\pm$. We can convert this single crossing change from $K(J)$ to $K_\pm$ into an immersed annulus in $S^3 \times I$ from $K(J)$ to $K_\pm$. Capping each of these annuli with a smooth slice disc for $K_\pm$ yields the desired immersed discs bounded by $K(J)$, each with a single singularity of different sign. $\square$

2.1. **Background results.** For $n \in \mathbb{N}$ and a knot K, we let $\Sigma_n(K)$ denote the nth cyclic branched cover of S^3 along K. To a knot K and a map $\chi\colon H_1(\Sigma_n(K)) \to \mathbb{Z}_q$ one can associate the Casson-Gordon signature $\sigma(K,\chi) \in \mathbb{Q}$ [CG86]. We avoid giving the technical definition of these invariants, noting only that they are defined in terms of the twisted intersection form of some 4-manifold and are notoriously difficult to compute precisely. We remark for those familiar with Casson-Gordon signatures that in the literature what we call $\sigma(K,\chi)$ is just $\sigma_1\tau(K,\chi)$ instead.

Our lower bound on the topological 4-genus of a knot comes from the following result of Gilmer.

Theorem 2.3 ([Gil82])**.** *Suppose that K is a knot with $g_4(K) \leq g$. Then there is a decomposition $H_1(\Sigma_2(K)) = A_1 \oplus A_2$ such that:*

(1) A_1 has a presentation with at most $2g$ generators.
(2) There is some $B \leq A_2$ with $|B|^2 = |A_2|$ such that for any prime power order $\chi\colon H_1(\Sigma(K)) \to \mathbb{Z}_q$, we have
$$|\sigma(K,\chi) + \sigma(K)| \leq 4g.$$

We remark for later that in our applications of Theorem 2.3 K will always be negative amphichiral and hence have $\sigma(K) = 0$.

Litherland proved a much more general formula for the Casson-Gordon invariants of satellite knots, but we will only need the following special case.

Theorem 2.4 ([Lit84])**.** *Suppose P is a pattern of winding number 0 described by an unknot η in the complement of $P(U)$. Let x denote the homology class of one of the lifts of η to $\Sigma_2(P(U))$. For any knot J, there is an isomorphism $\alpha\colon H_1(\Sigma_2(P(J))) \to H_1(\Sigma_2(P(U)))$ such that for any $\chi\colon H_1(\Sigma_2(P(U))) \to \mathbb{Z}_q$ we have*
$$\sigma(P(J), \chi \circ \alpha) = \sigma(P(U), \chi) + 2\sigma_J(\omega_q^{\chi(x)}),$$
where $\omega_q = e^{2\pi i/q}$ and σ_J denotes the Tristram-Levine signature function.

As well as the knot invariant $\sigma(K,\chi)$, Casson-Gordon introduced a signature invariant $\sigma(M,\phi)$ associated to a 3-manifold M and a character $\phi\colon H_1(M) \to \mathbb{Z}_q$. We will need a formula due to Cimasoni-Florens for the Casson-Gordon signature of a 3-manifold in terms of the colored signature function of a surgery link. Although this result is proved in much more generality, we state it only for the case of interest: when M is obtained by surgery on a Hopf link. We thereby avoid going into the technical details of the definition of the colored signature function, noting only for the experts that the cell complex consisting of 2 discs meeting in a single arc and bounded by the Hopf link is a C-complex in the sense of [CF08], and the contractibility of this complex immediately implies that the colored signature function of the Hopf link is identically zero.

Theorem 2.5. [CF08, Theorem 6.7] *Suppose that a 3-manifold M is obtained by surgery on a Hopf link L with linking matrix $\Lambda = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$. Let q be prime and $\chi\colon H_1(M) \to \mathbb{Z}_q$ be a character such that the two meridians μ_1, μ_2 of L are sent to nonzero elements of $\mathbb{Z}_q$. For $i = 1, 2$ let $n_i \in \{1, \dots, q-1\}$ be the unique value satisfying $n_i \equiv \chi(\mu_i) \mod q$. Then*
$$\sigma(M,\chi) = -1 - sign(\Lambda) + \frac{2}{q^2}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}^T \cdot \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \cdot \begin{bmatrix} q - n_1 \\ q - n_2 \end{bmatrix}$$

2.2. **Proof of Theorem 1.1.** We now apply Theorems 2.4 and 2.5 to obtain a formula for the Casson-Gordon signatures of K_J in terms of the Tristram-Levine signatures of J.

Example 2.6. Let K_0 denote the figure-eight knot. Note that $K(J)$ is obtained from K_0 by two infections along curves η_1 and η_2, as depicted in Figure 3.

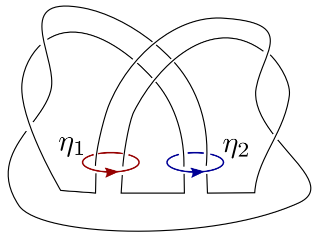

FIGURE 3. The knot $K(J)$ is an iterated satellite of the figure-eight knot.

By twice applying Theorem 2.4, we see that for any knot J there is an isomorphism $\alpha\colon H_1(\Sigma_2(K(J))) \to H_1(\Sigma_2(K_0))$ such that for any character $\chi\colon H_1(\Sigma_2(K_0)) \to \mathbb{Z}_q$ we have

$$\sigma(K(J), \alpha \circ \chi) = \sigma(K_0, \chi) + 2\sigma_J(\omega_q^{\chi(\widetilde{\eta}_1)}) + 2\sigma_{\overline{J}}(\omega_q^{\chi(\widetilde{\eta}_2)}) = \sigma(K_0, \chi) + 2\sigma_J(\omega_q^{\chi(\widetilde{\eta}_1)}) - 2\sigma_J(\omega_q^{\chi(\widetilde{\eta}_2)})$$

Since both η_i curves are disjoint from the usual genus one Seifert surface for K_0, we can apply Akbulut-Kirby's algorithm of [AK80] to obtain the following surgery diagram for $\Sigma_2(K_0)$, with lifts of η_1 and η_2 as indicated. (Note that we have only depicted one lift of each curve, since that is all we need to apply Theorem 2.5.) The first homology of $\Sigma_2(K_0)$ is generated by the meridians of

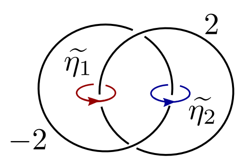

FIGURE 4. A surgery diagram L for $\Sigma_2(K_0)$.

the components of L, which are isotopic to $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$. The relations are given by the rows of the linking-framing matrix, and are

$$-2[\widetilde{\eta}_2] + [\widetilde{\eta}_1] = 0 \text{ and } [\widetilde{\eta}_2] + 2[\widetilde{\eta}_1] = 0.$$

Some quick simplifications give us that $H_1(\Sigma_2(K_0)) \cong \mathbb{Z}_5$, generated by $a := [\widetilde{\eta}_2]$ and such that $[\widetilde{\eta}_1] = 2[\widetilde{\eta}_2]$. Therefore, for any character $\chi\colon H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$ we have that

$$\sigma(K_J, \chi \circ \alpha) = \sigma(K_0, \chi) + \sigma_J(\omega_5^{2\chi(a)}) - \sigma_J(\omega_5^{\chi(a)}). \tag{1}$$

We can also use the surgery diagram of Figure 4 to bound $|\sigma(K_0, \chi)|$. For $j \in \mathbb{Z}_5$, define $\chi_j\colon H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$ to be the map with $\chi_j(x) = j$. Observe that $\chi_1([\widetilde{\eta}_1]) = 2$ and $\chi_2([\widetilde{\eta}_1]) = 4$. Therefore, Theorem 2.5 gives us that

$$\sigma(\Sigma_2(K_0), \chi_1) = -1 - 0 + \frac{2}{25}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = -1 + \frac{30}{25} = 1/5$$

and

$$\sigma(\Sigma_2(K_0), \chi_2) = -1 - 0 + \frac{2}{25}\begin{bmatrix} 2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -1 + \frac{20}{25} = -1/5.$$

Moreover, basic properties of Casson-Gordon signatures (or reapplying Theorem 2.5) imply that $\sigma(\Sigma_2(K_0), \chi_3) = \sigma(\Sigma_2(K_0), \chi_2)$, $\sigma(\Sigma_2(K_0), \chi_4) = \sigma(\Sigma_2(K_0), \chi_1)$, and $\sigma(\Sigma_2(K_0), \chi_0) = 0$.

Since $H_1(\Sigma_2(K_0)) \cong \mathbb{Z}_5$ is cyclic, for any character $\chi\colon H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$ we have by [CG86, Lemma 3 and Theorem 4] that

$$|\sigma(K_0, \chi) - \sigma(\Sigma_2(K_0), \chi)| \leq 1.$$

Therefore, we conclude that for any $\chi\colon H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$ we have $|\sigma(K_0, \chi)| < 2$.

We are now ready to prove the following and obtain Theorem 1.1 as a consequence.

Proposition 2.7. *Fix $g \in \mathbb{N}$. For $i \in \mathbb{N}$ define $J_i = \#^{m_i} T_{2,5}$, where $m_i = 2^{2i+1}g$. Now, for $k \in \mathbb{N}$ define $K^k := \#_{i=1}^{2g+2} K(J_{k(2g+2)+i})$. Then $S = \{K^k\}_{k\in\mathbb{N}}$ is a collection of algebraically slice knots such that any nontrivial sum $K = \#_{j=1}^{n} K^{k_j}$ satisfies the conclusions of Theorem 1.1.*

Proof. Observe that for any choice of J, the knot $K(J)$ shares a Seifert form with K_0. Therefore, each K^k shares a Seifert form with the slice knot $\#_{i=1}^{2g+2} K_0$, and hence is algebraically slice.

Now let $K = \#_{j=1}^{n} K^{k_j}$ be a nontrivial sum of elements of S. We can and do assume that $k_1 < k_2 < \cdots < k_n$. Since conditions (1) and (2) of Theorem 1.1 are preserved under connected sum, it only remains to verify condition (3).

So suppose for a contradiction that $g_4(K) \leq g$ and hence that there exists a decomposition $H_1(\Sigma_2(K)) \cong A_1 \oplus A_2$ and a subgroup $B \leq A_2$ satisfying the conclusions of Theorem 2.3. Let

$$\beta\colon H_1(\Sigma_2(K)) \to \bigoplus_{j=1}^{n}\left(\bigoplus_{i=1}^{2g+2} H_1(\Sigma_2(K(J_{k(2g+2)+i})))\right) \to \bigoplus_{j=1}^{n}\left(\bigoplus_{i=1}^{2g+2} H_1(\Sigma_2(K_0))\right)$$

denote the isomorphism (coming from Theorem 2.4 together with the additivity of Casson-Gordon signatures with respect to connected sum [Lit84]) satisfying

$$\begin{aligned}
\sigma\left(K, \beta \circ \left((\chi_i^j)_{i=1}^{2g+2}\right)_{j=1}^{n}\right) &= \sum_{j=1}^{m} \sigma(K^{k_j}, (\chi_i^j)_{i=1}^{2g+2}) \\
&= \sum_{j=1}^{n}\left(\sum_{i=1}^{2g+2} \sigma(K(J_{k_j(2g+2)+i}), \chi_i^j)\right) \\
&= \sum_{j=1}^{n}\left(\sum_{i=1}^{2g+2} \sigma(K_0, \chi_i^j) + 2\sigma_{J_{k_j(2g+2)+i}}(\omega_5^{2\chi_i^j(a)}) - 2\sigma_{J_{k_j(2g+2)+i}}(\omega_5^{\chi_i^j(a)})\right),
\end{aligned}$$

where in the last equality we use Equation 1 of Example 2.6.

Since $H_1(\Sigma_2(K)) \cong \mathbb{Z}_5^{m(2g+2)}$ and A_1 has a presentation with at most $2g$ generators, we have that A_1 is isomorphic to $\mathbb{Z}_5^j$ for some $j \leq 2g$. Therefore A_2 is isomorphic to $\mathbb{Z}_5^{n(2g+2)-j}$ and B is isomorphic to $\mathbb{Z}_5^{n(g+1)-j/2}$. So $A_1 \oplus B \cong \mathbb{Z}_5^{j'}$ for

$$j' = n(g+1) + j/2 \leq n(g+1) + g < n(2g+2)$$

and there exists a nonzero character $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_5$ that vanishes on $A_1 \oplus B$.

The rest of the proof consists of showing that $|\sigma(K,\chi)| > 4g$, using only our definition of K and the hypothesis that χ is not identically zero. Let

$$\left((\chi_i^j)_{i=1}^{2g+2}\right)_{j=1}^{n} := \chi \circ \beta^{-1}\colon \bigoplus_{j=1}^{n}\left(\bigoplus_{i=1}^{2g+2} H_1(\Sigma_2(K_0))\right) \to \mathbb{Z}_5.$$

Since χ is nontrivial, there exists some j such that $(\chi_i^j)_{i=1}^{2g+1}$ is not identically zero. Let j_0 be the maximal such j and i_0 be the maximal i such that $\chi_i^{j_0}$ is nonzero. Let $\ell = k_{j_0}(2g+2) + i_0$. The following algebraic manipulations show that $\sigma(K(J_\ell), \chi_{i_0}^{j_0})$ so dominates the other terms that could contribute to $\sigma(K,\chi)$ that we have as desired that $|\sigma(K,\chi)| > 4g$.

Recalling that $J_i = \#^{m_i} T_{2,5}$, where $m_i = 2^{2i+1}g$, we have by the additivity of Tristram-Levine signatures under connected sum that $\sigma_{J_i}(\omega_5) = \sigma_{J_i}(\omega_5^4) = -2^{2i+2}g$ and $\sigma_{J_i}(\omega_5^2) = \sigma_{J_i}(\omega_5^3) = -2^{2i+3}g$ (see KnotInfo [LM20] for the Tristram-Levine signature function of $T(2,5)$.) Applying

Equation 1 from Example 2.6, we see that for any i and any nonzero character $\rho\colon H_1(\Sigma_2(K(J_i))) \to \mathbb{Z}_5$ we have that

$$2^{2i+3}g - 2 \leq |\sigma(K(J_i),\rho)| = |\sigma(K_0,\rho) \pm (2\sigma_{J_i}(\omega_5) - 2\sigma_{J_i}(\omega_5^2))| \leq 2^{2i+3}g + 2 \tag{2}$$

Note that here and in the rest of the proof, we suppress the identification of each $H_1(\Sigma_2(K(J_i)))$ with $H_1(\Sigma_2(K_0))$.

Observe that the set of natural numbers

$$\{k_{j_0}(2g+2) + i : 1 \leq i \leq i_0 - 1\} \cup \bigcup_{j=1}^{j_0-1} \{k_j(2g+2)+i) : 1 \leq i \leq 2g+2\} \tag{3}$$

is a subset of $\{1,\ldots,\ell-1\}$, recalling that $\ell = k_{j_0}(2g+2)+i_0$. We therefore have that

$$\begin{aligned}
|\sigma(K,\chi)| &= \left|\sum_{j=1}^{n}\sum_{i=1}^{2g+2}\sigma(K(J_{k_j(2g+2)+i}),\chi_i^j)\right| \\
&= \left|\sigma(K(J_\ell),\chi_{i_0}^{j_0}) + \sum_{i=1}^{i_0-1}\sigma(K(J_{k_{j_0}(2g+2)+i}),\chi_i^{j_0}) + \sum_{j=1}^{j_0-1}\sum_{i=1}^{2g+2}\sigma(K(J_{k_j(2g+2)+i}),\chi_i^j)\right| \\
&\geq \left|\sigma(K(J_\ell),\chi_{i_0}^{j_0})\right| - \sum_{i=1}^{i_0-1}\left|\sigma(K(J_{k_{j_0}(2g+2)+i}),\chi_i^{j_0})\right| - \sum_{j=1}^{j_0-1}\sum_{i=1}^{2g+2}\left|\sigma(K(J_{k_j(2g+2)+i}),\chi_i^j)\right| \\
&\geq (2^{2\ell+3}g - 2) - \sum_{k=1}^{\ell-1}(2^{2k+3}g+2) =: (*)
\end{aligned}$$

where in the last inequality we use our observation from Equation 3 together with Equation 2. Some algebraic simplification yields that

$$(*) = 8g\left(2^{2\ell} - \sum_{k=1}^{\ell-1}2^{2k}\right) - 2\ell = (g/3)(2^{2\ell+3} - 32) - 2\ell.$$

Now, note that since $\ell > 2g+2 \geq 4$ we have that $2\ell+3 > 11$ and so certainly $2^{2\ell+3} - 32 > 2^{2\ell+2}$. Therefore

$$|\sigma(K,\chi)| \geq (*) > (g/3)2^{2\ell+2} - 2\ell > 2^{2\ell} - 2\ell.$$

Finally, we observe that for any $x > 2$ we have $2^{2x} - 2x > 2x$, since letting $f(x) = 2^{2x} - 4x$ we see that $f'(x) = \ln(4)2^{2x} - 4$ is positive for all $x \geq 1$ and $f(2) = 8$. Therefore

$$|\sigma(K,\chi)| > 2\ell > 4g+4 > 4g,$$

as desired. $\square$

Remark 2.8. The examples of Proposition 2.7 are far from the only knots satisfying the conclusions of Theorem 1.1. One could vary the base knot, for example by choosing $\{a_i\}_{i\geq 0}$ to be natural numbers such that $\{4a_i^2+1\}_{i\in\mathbb{N}}$ consists of pairwise relatively prime numbers. (This is easily accomplished by e.g. letting $a_0 = 1$ and $a_k = \prod_{i=1}^{k-1}(4a_i^2+1)$ for $k \geq 1$.) Now, let K_i be the 2-bridge knot corresponding to the rational number $\frac{4a_i^2+1}{2a_i}$, noting that indeed K_0 is the figure-eight knot. Choose $\{p_i\}_{i\geq 0}$ to be primes dividing $4a_i^2+1$, noting that by our choice of a_i we have that p_i divides $4a_j^2+1$ if and only if $j = i$. By taking connected sums of K_{a_i} analogously infected with large connected sums of T_{2,p_i} and $-T_{2,p_i}$, we can essentially repeat the arguments of Proposition 2.7 and obtain many more linearly independent knots satisfying the conclusions of Theorem 1.1.

References

[AK80] Selman Akbulut and Robion Kirby. Branched covers of surfaces in 4-manifolds. *Math. Ann.*, 252(2):111–131, 1979/80.

[CF08] David Cimasoni and Vincent Florens. Generalized Seifert surfaces and signatures of colored links. *Trans. Amer. Math. Soc.*, 360(3):1223–1264, 2008.

[CG86] Andrew Casson and Cameron Gordon. Cobordism of classical knots. In *À la recherche de la topologie perdue*, volume 62 of *Progr. Math.*, pages 181–199. Birkhäuser Boston, Boston, MA, 1986. With an appendix by P. M. Gilmer.

[FMPC19] Peter Feller, Allison N. Miller, and Juanita Pinzón-Caicedo. A note on the topological slice genus of satellite knots. *preprint arXiv:1908.03760*, 2019.

[Gil82] Patrick M. Gilmer. On the slice genus of knots. *Invent. Math.*, 66(2):191–197, 1982.

[Hau78] J. C. Hausmann, editor. *Problem list, from Knot theory: (Proc. Sem., Plans-sur-Bex, 1977)*, volume 685. Springer-Verlag Berlin Heidelberg, 1978.

[HKJPS20] Jennifer Hom, Sungkyung Kang, JungHwan Park, and Matthew Stoffregen. Linear independence of rationally slice knots. *preprint arXiv:2011.07659*, 2020.

[HW16] Jennifer Hom and Zhongtao Wu. Four-ball genus bounds and a refinement of the Ozváth-Szabó tau invariant. *J. Symplectic Geom.*, 14(1):305–323, 2016.

[JZ20] András Juhász and Ian Zemke. New Heegaard Floer slice genus and clasp number bounds. *preprint arXiv:2007.07106*, 2020.

[Kaw09] Akio Kawauchi. Rational-slice knots via strongly negative-amphicheiral knots. *Commun. Math. Res.*, 25(2):177–192, 2009.

[KL99] Paul Kirk and Charles Livingston. Twisted knot polynomials: inversion, mutation and concordance. *Topology*, 38(3):663–671, 1999.

[KM13] P. B. Kronheimer and T. S. Mrowka. Gauge theory and Rasmussen's invariant. *J. Topol.*, 6(3):659–674, 2013.

[Lev69] J. Levine. Knot cobordism groups in codimension two. *Comment. Math. Helv.*, 44:229–244, 1969.

[Lit84] Richard Litherland. Cobordism of satellite knots. In *Four-manifold theory (Durham, N.H., 1982)*, volume 35 of *Contemp. Math.*, pages 327–362. Amer. Math. Soc., Providence, RI, 1984.

[Liv99] Charles Livingston. Order 2 algebraically slice knots. In *Proceedings of the Kirbyfest (Berkeley, CA, 1998)*, volume 2 of *Geom. Topol. Monogr.*, pages 335–342. Geom. Topol. Publ., Coventry, 1999.

[LL19] Lukas Lewark and Andrew Lobb. Upsilon-like concordance invariants from sl_n knot cohomology. *Geom. Topol.*, 23(2):745–780, 2019.

[LM20] Charles Livingston and Allison H. Moore. KnotInfo: Table of Knot Invariants. *http://www.indiana.edu/ knotinfo*, November 16, 2020.

[Lob09] Andrew Lobb. A slice genus lower bound from sl(n) Khovanov-Rozansky homology. *Adv. Math.*, 222(4):1220–1276, 2009.

[Miy94] Katura Miyazaki. Nonsimple, ribbon fibered knots. *Trans. Amer. Math. Soc.*, 341(1):1–44, 1994.

[OS03] Peter Ozsváth and Zoltán Szabó. Knot Floer homology and the four-ball genus. *Geom. Topol.*, 7:615–639, 2003.

[OSS17] Peter S. Ozsváth, András I. Stipsicz, and Zoltán Szabó. Concordance homomorphisms from knot Floer homology. *Adv. Math.*, 315:366–426, 2017.

[Ras10] Jacob Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010.

[Tri69] A. G. Tristram. Some cobordism invariants for links. *Proc. Cambridge Philos. Soc.*, 66:251–264, 1969.

[Wu09] Hao Wu. On the quantum filtration of the Khovanov-Rozansky cohomology. *Adv. Math.*, 221(1):54–139, 2009.

Department of Mathematics, Rice University, Houston, Texas, USA
Email address: `allison.miller@rice.edu`