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FUČIK SPECTRUM WITH WEIGHTS AND EXISTENCE OF SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

NSOKI MAVINGA, QUINN A. MORRIS, STEPHEN B. ROBINSON

In Memory of John W. Neuberger

ABSTRACT. We consider the boundary value problem

$$\begin{aligned} -\Delta u + c(x)u &= \alpha m(x)u^+ - \beta m(x)u^- + f(x, u), & x \in \Omega, \\ \frac{\partial u}{\partial \eta} + \sigma(x)u &= \alpha \rho(x)u^+ - \beta \rho(x)u^- + g(x, u), & x \in \partial\Omega, \end{aligned}$$

where $(\alpha, \beta) \in \mathbb{R}^2$, $c, m \in L^\infty(\Omega)$, $\sigma, \rho \in L^\infty(\partial\Omega)$, and the nonlinearities f and g are bounded continuous functions. We study the asymmetric (Fučík) spectrum with weights, and prove existence theorems for nonlinear perturbations of this spectrum for both the resonance and non-resonance cases. For the resonance case, we provide a sufficient condition, the so-called generalized Landesman-Lazer condition, for the solvability. The proofs are based on variational methods and rely strongly on the variational characterization of the spectrum.

1. INTRODUCTION

We consider the partial differential equation

$$\begin{aligned} -\Delta u + c(x)u &= m(x)[\alpha u^+ - \beta u^-], & x \in \Omega, \\ \frac{\partial u}{\partial \eta} + \sigma(x)u &= \rho(x)[\alpha u^+ - \beta u^-], & x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Delta z := \nabla \cdot \nabla z$, $\frac{\partial}{\partial \eta}$ is the outward normal derivative, $(\alpha, \beta) \in \mathbb{R}^2$ are parameters, and $c, m \in L^\infty(\Omega)$, $\sigma, \rho \in L^\infty(\partial\Omega)$ with $c(x), m(x) \geq 0$ almost everywhere in Ω , $\sigma(x), \rho(x) \geq 0$ almost everywhere in $\partial\Omega$,

$$\int c(x) dx + \oint \sigma(x) dx > 0 \quad \text{and} \quad \int m(x) dx + \oint \rho(x) dx > 0,$$

where \int denotes the (volume) integral on Ω and \oint denotes the (surface) integral on $\partial\Omega$. Throughout this paper we assume that Ω is a bounded domain in \mathbb{R}^n ($n \geq 2$) with smooth boundary $\partial\Omega$.

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We are interested in the Fučík spectrum, namely,

$$\Sigma := \{(\alpha, \beta) \in \mathbb{R}^2 : (1.1) \text{ has a non-trivial solution}\}$$

and our first main result provides a variational characterization of a curve in Σ .

As an application of the variational characterization we consider

$$\begin{aligned} -\Delta u + c(x)u &= m(x)[\alpha u^+ - \beta u^-] + f(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \sigma(x)u &= \rho(x)[\alpha u^+ - \beta u^-] + g(x, u) & \text{on } \partial\Omega, \end{aligned} \quad (1.2)$$

i.e. a nonlinear perturbation of (1.1). We assume nonlinearities of the form $f(x, u) := m(x)\tilde{f}(u)$ and $g(x, u) := \rho(x)\tilde{g}(u)$, where $\tilde{f}, \tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$ are bounded continuous functions. We prove existence theorems for the non-resonance case, $(\alpha, \beta) \notin \Sigma$, and the resonance case, $(\alpha, \beta) \in \Sigma$. For the resonance case we assume a generalized Landesman-Lazer condition as in [4] and [8].

Our methods are built on the results in [7, 4, 8]. Section 2 provides a brief summary of the function spaces and the variational setting. In Section 3, we prove the variational characterization of a curve in Σ using a Hilbert space reduction method as in [2, 4, 8]. Section 4 contains the existence theorem for the non-resonance case. Section 5 contains the existence theorem for the resonance case.

2. CHARACTERIZATION OF THE FUČIK SPECTRUM

2.1. Variational preliminaries. Define the (c, σ) -inner product $\langle \cdot, \cdot \rangle_{(c, \sigma)} : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ by

$$\langle u, v \rangle_{(c, \sigma)} = \int \nabla u \cdot \nabla v + \int c(x)uv + \oint \sigma(x)uv,$$

with the associated norm denoted by $\|u\|_{(c, \sigma)}$. This norm is equivalent to the standard $H^1(\Omega)$ -norm. Set

$$\langle u, v \rangle_{(m, \rho)} = \int m(x)uv + \oint \rho(x)uv, \quad \|u\|_{(m, \rho)}^2 := \int m(x)u^2 + \oint \rho(x)u^2,$$

for $u, v \in H^1(\Omega)$.

Let $V_{(m, \rho)} = \{u \in H^1(\Omega) : \|u\|_{(m, \rho)} = 0\}$, and let $H_{(m, \rho)}^1 = V_{(m, \rho)}^\perp$ be the orthogonal complement with respect to the (c, σ) inner product. Then $H^1(\Omega) = H_{(m, \rho)}^1 \oplus V_{(m, \rho)}$ (see [7]) and it further follows that $H_{(m, \rho)}^1$ and $V_{(m, \rho)}$ are (m, ρ) orthogonal. We will also make use of the norm $\|\cdot\|_{(c, \sigma)}$ on $H^1(\Omega)$ and $\|\cdot\|_{(m, \rho)}$ on $H_{(m, \rho)}^1$.

We also provide an alternate characterization of $V_{(m, \rho)}$ from [7]: taking $\Omega(m) := \{x \in \Omega : m(x) > 0\}$ and $\partial\Omega(\rho) := \{x \in \partial\Omega : \rho(x) > 0\}$, we have

$$V_{(m, \rho)} = \{u \in H^1(\Omega) : u = 0 \text{ a.e in } \Omega(m) \text{ and } \Gamma u = 0 \text{ a.e in } \partial\Omega(\rho)\}, \quad (2.1)$$

where Γ is the trace operator on $\partial\Omega$.

Consider the functional $J : H^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$J_{\alpha, \beta}(u) = \frac{1}{2}[\|u\|_{(c, \sigma)}^2 - \alpha\|u^+\|_{(m, \rho)}^2 - \beta\|u^-\|_{(m, \rho)}^2]. \quad (2.2)$$

Then

$$J'_{\alpha, \beta}(u) \cdot v = \langle u, v \rangle_{(c, \sigma)} - \alpha\langle u, v \rangle_{(m, \rho)} + (\beta - \alpha)\langle u^-, v \rangle_{(m, \rho)}. \quad (2.3)$$

We note that critical points of $J_{\alpha, \beta}$ are weak solutions of (1.1).

We begin with a lemma on the nature of the Fučík eigenfunctions.

Lemma 2.1. *Every Fučík eigenfunction ψ is contained in $H^1_{(m,\rho)}$.*

Proof. Assume to the contrary that $\psi = u + v$, where $u \in H^1_{(m,\rho)}$, $v \in V_{(m,\rho)}$, and v is nonzero on a set of positive measure. Then

$$\begin{aligned} 0 &= J'_{\alpha,\beta}(\psi) \cdot v \\ &= \langle u + v, v \rangle_{(c,\sigma)} - \alpha \langle u + v, v \rangle_{(m,\rho)} + (\beta - \alpha) \langle (u + v)^-, v \rangle_{(m,\rho)} \\ &= \|v\|^2_{(c,\sigma)}, \end{aligned}$$

because of the alternate characterization of $V_{(m,\rho)}$ in (2.1). Hence, $v = 0$ a.e. which contradicts our assumption. So all Fučík eigenfunctions are in $H^1_{(m,\rho)}$. \square

2.2. Trivial curves. It is known (see [7]) that the problem

$$\begin{aligned} -\Delta u + c(x)u &= \mu m(x)u, & x \in \Omega, \\ \frac{\partial u}{\partial \eta} + \sigma(x)u &= \mu \rho(x)u, & x \in \partial\Omega, \end{aligned}$$

has a simple first eigenvalue $\mu_1 > 0$ with associated eigenfunction ϕ_1 which is of one sign in $\bar{\Omega}$. Therefore $\phi_1^+ = \phi_1$ and $\phi_1^- = 0$, so that

$$-\Delta \phi_1 + c(x)\phi_1 = \mu_1 m(x)\phi_1 = m(x)[\mu_1 \phi_1^+ - \beta \phi_1^-]$$

for any $\beta \in \mathbb{R}$, and similarly

$$\frac{\partial \phi_1}{\partial \eta} + \sigma(x)\phi_1 = \rho(x)[\mu_1 \phi_1^+ - \beta \phi_1^-]$$

for any $\beta \in \mathbb{R}$. Therefore

$$\mathcal{C}_0 := \{(\mu_1, \beta) : \beta \in \mathbb{R}\} \subset \Sigma.$$

A similar argument will show that

$$\mathcal{C}'_0 := \{(\alpha, \mu_1) : \alpha \in \mathbb{R}\} \subset \Sigma.$$

The curves \mathcal{C}_0 and \mathcal{C}'_0 are depicted in Figure 1.

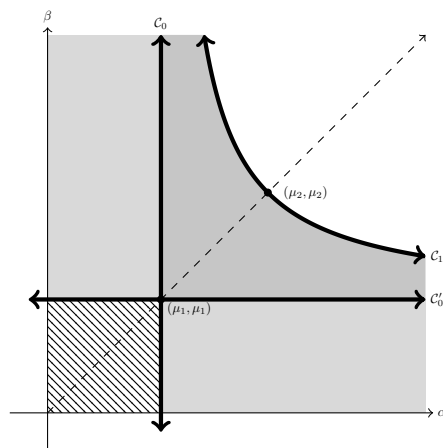


FIGURE 1. Trivial and first Fučík curves

Lemma 2.2.

$$\Sigma \cap \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha < \mu_1 \text{ or } \beta < \mu_1\} \cap (\mathcal{C}_0 \cup \mathcal{C}'_0)^C = \emptyset$$

Proof. Let $\alpha < \mu_1$ and $\beta \neq \mu_1$. Assume that $(\alpha, \beta) \in \Sigma$ and let $\psi \in H^1_{(m,\rho)}$ be a Fučík eigenfunction associated to (α, β) . Then

$$0 = J'_{\alpha,\beta}(\psi) \cdot \psi^+ = \|\psi^+\|_{(c,\sigma)}^2 - \alpha \|\psi^+\|_{(m,\rho)}^2 \geq (\mu_1 - \alpha) \|\psi^+\|_{(m,\rho)}^2.$$

So, since $\alpha < \mu_1$, it follows that $\|\psi^+\|_{(m,\rho)}^2 = 0$, which implies that $\psi^+ = 0$ almost everywhere. Hence, $\psi = -\psi^-$, and hence ψ is a non-positive Steklov eigenfunction. So ψ satisfies

$$\begin{aligned} -\Delta\psi + c(x)\psi &= m(x)\beta\psi; & x \in \Omega, \\ \frac{\partial\psi}{\partial\eta} + \sigma(x)\psi &= \rho(x)\beta\psi; & x \in \partial\Omega. \end{aligned}$$

But if ψ is a non-sign-changing solution, then $\beta = \mu_1$, a contradiction. Hence $(\alpha, \beta) \notin \Sigma$.

If $\beta < \mu_1$ and $\alpha \neq \mu_1$, the argument proceeds similarly by examining the expression $J'_{\alpha,\beta}(\psi) \cdot \psi^-$. \square

2.3. Higher curves. In what follows, we will consider the case $\mu_k < \alpha < \mu_{k+1}$ and $\alpha < \beta$. If $(\alpha, \beta) \in \Sigma$, then $(\beta, \alpha) \in \Sigma$, and therefore, it suffices to only consider the case $\alpha < \beta$. The first curve \mathcal{C}_1 is depicted in Figure 1.

We split the space $H^1_{(m,\rho)} = X_k \oplus Y_k$ where $X_k = \text{span}\{\phi_1, \phi_2, \dots, \phi_k\}$ and $Y_k = \text{span}\{\phi_{k+1}, \phi_{k+2}, \dots\}$. We further define $Y = Y_k \oplus V_{(m,\rho)}$ so that $H^1 = X_k \oplus Y$. We begin with an estimate which will be crucial for several lemmas later.

Lemma 2.3. *Let $(\alpha_i, \beta_i) \in \mathbb{R}^2$ for $i = 1, 2$ satisfy the previous hypotheses, and let $s_i = \beta_i - \alpha_i$. Let $x_i \in X_k$ and $y_i \in Y$ for $i = 1, 2$. Then*

$$\begin{aligned} & (J'_{\alpha_2,\beta_2}(x_2 + y_2) - J'_{\alpha_1,\beta_1}(x_1 + y_1)) \cdot (x_2 - x_1) \\ & \leq -\delta \|x_2 - x_1\|_{(c,\sigma)}^2 + s_2 (\|x_2 - x_1\|_{(m,\rho)} + \|y_2 - y_1\|_{(m,\rho)}) \|y_2 - y_1\|_{(m,\rho)} \\ & \quad + |\alpha_2 - \alpha_1| \|x_1\|_{(m,\rho)} \|x_2 - x_1\|_{(m,\rho)} + |s_2 - s_1| \|x_1 + x_2\|_{(m,\rho)} \|x_2 - x_1\|_{(m,\rho)}, \end{aligned}$$

where $\delta = \frac{\alpha_2}{\mu_k} - 1$.

Proof. First we show that

$$\begin{aligned} & J'_{\alpha_i,\beta_i}(x_i + y_i)(x_2 - x_1) \\ & = \langle x_i + y_i, x_2 - x_1 \rangle_{(c,\sigma)} - \alpha_i \langle x_i + y_i, x_2 - x_1 \rangle_{(m,\rho)} + s_i \langle (x_i + y_i)^-, x_2 - x_1 \rangle_{(m,\rho)} \\ & = \langle x_i, x_2 - x_1 \rangle_{(c,\sigma)} - \alpha_i \langle x_i, x_2 - x_1 \rangle_{(m,\rho)} \\ & \quad + s_i \langle (x_i + y_i)^-, x_2 - x_1 \rangle_{(m,\rho)}, \end{aligned}$$

by the (c, σ) - and (m, ρ) -orthogonality of X_k and Y . Then utilizing the previous expression, we have

$$\begin{aligned}
 & (J'_{\alpha_2, \beta_2}(x_2 + y_2) - J'_{\alpha_1, \beta_1}(x_1 + y_1)) \cdot (x_2 - x_1) \\
 &= \|x_2 - x_1\|_{(c, \sigma)}^2 - \langle \alpha_2 x_2 - \alpha_1 x_1, x_2 - x_1 \rangle_{(m, \rho)} \\
 &\quad + \langle s_2(x_2 + y_2)^- - s_1(x_1 + y_1)^-, x_2 - x_1 \rangle_{(m, \rho)} \\
 &= \|x_2 - x_1\|_{(c, \sigma)}^2 - \alpha_2 \|x_2 - x_1\|_{(m, \rho)}^2 - (\alpha_2 - \alpha_1) \langle x_1, x_2 - x_1 \rangle_{(m, \rho)} \\
 &\quad + s_2 \langle (x_2 + y_2)^- - (x_1 + y_1)^-, x_2 - x_1 \rangle_{(m, \rho)} \\
 &\quad + (s_2 - s_1) \langle (x_1 + y_1)^-, x_2 - x_1 \rangle_{(m, \rho)}
 \end{aligned} \tag{2.4}$$

By the variational characterization of μ_k and the definition of X_k , we have that

$$\|x_2 - x_1\|_{(c, \sigma)}^2 - \alpha_2 \|x_2 - x_1\|_{(m, \rho)}^2 \leq \left(1 - \frac{\alpha_2}{\mu_k}\right) \|x_2 - x_1\|_{(c, \sigma)}^2 = -\delta \|x_2 - x_1\|_{(c, \sigma)}^2.$$

Since $f(t) = t^-$ is non-increasing, we have that $v_1^- - v_2^-$ and $v_1 - v_2$ have opposite sign for all $v_1, v_2 \in H^1$. Furthermore, $|f(t_2) - f(t_1)| \leq |t_2 - t_1|$. Hence,

$$\begin{aligned}
 & s_2 \langle (x_2 + y_2)^- - (x_1 + y_1)^-, x_2 - x_1 \rangle_{(m, \rho)} \\
 &= s_2 \langle (x_2 + y_2)^- - (x_1 + y_1)^-, (x_2 + y_2) - (x_1 + y_1) \rangle_{(m, \rho)} \\
 &\quad + s_2 \langle (x_2 + y_2)^- - (x_1 + y_1)^-, y_1 - y_2 \rangle_{(m, \rho)} \\
 &\leq s_2 \langle |(x_2 + y_2)^- - (x_1 + y_1)^-|, |y_1 - y_2| \rangle_{(m, \rho)} \\
 &\leq s_2 \langle |(x_2 + y_2) - (x_1 + y_1)|, |y_1 - y_2| \rangle_{(m, \rho)} \\
 &= s_2 \langle |(x_2 - x_1) - (y_2 - y_1)|, |y_1 - y_2| \rangle_{(m, \rho)} \\
 &\leq s_2 (\|x_2 - x_1\|_{(m, \rho)} + \|y_2 - y_1\|_{(m, \rho)}) \|y_2 - y_1\|_{(m, \rho)}.
 \end{aligned}$$

Using Hölder's inequality, we estimate the remaining two terms as

$$|(\alpha_2 - \alpha_1) \langle x, x_2 - x_1 \rangle_{(m, \rho)}| \leq |\alpha_2 - \alpha_1| \|x_1\|_{(m, \rho)} \|x_2 - x_1\|_{(m, \rho)}$$

and

$$|(s_2 - s_1) \langle (x_1 + y_1)^-, x_2 - x_1 \rangle_{(m, \rho)}| \leq |s_2 - s_1| \|x_1 + x_2\|_{(m, \rho)} \|x_2 - x_1\|_{(m, \rho)}.$$

Combining the previous estimates into (2.4) yields the desired result. \square

Lemma 2.4. *For a fixed $y \in Y$, $J_{\alpha, \beta}(x + y)$ is concave on X_k and moreover, for any $x_1, x_2 \in X_k$,*

$$(J'_{\alpha, \beta}(x_2 + y) - J'_{\alpha, \beta}(x_1 + y)) \cdot (x_2 - x_1) \leq -\delta \|x_2 - x_1\|_{(c, \sigma)}^2.$$

Proof. Take $y_1 = y_2 = y$, $\alpha_1 = \alpha_2 = \alpha$, and $\beta_1 = \beta_2 = \beta$ in Lemma 2.3. Then $s_1 = s_2 = \beta - \alpha$, and the inequality reduces to

$$(J'_{\alpha, \beta}(x_2 + y) - J'_{\alpha, \beta}(x_1 + y)) \cdot (x_2 - x_1) \leq -\delta \|x_2 - x_1\|_{(c, \sigma)}^2,$$

as desired. If we further set $x_1 = 0$ and $x_2 = x$, we observe that

$$(J'_{\alpha, \beta}(x + y) - J'_{\alpha, \beta}(y)) \cdot x \leq -\delta \|x\|_{(c, \sigma)}^2,$$

and hence $J_{\alpha, \beta}(x + y)$ is concave on X_k . \square

Since $J_{\alpha,\beta}$ is concave on X_k , for any fixed $y \in Y$, we define $r_{\alpha,\beta}(y) \in X_k$ to be the unique maximizer of $J_{\alpha,\beta}$ restricted to $X_k + y$, namely

$$J_{\alpha,\beta}(r_{\alpha,\beta}(y) + y) = \max_{x \in X_k} J_{\alpha,\beta}(x + y). \quad (2.5)$$

We now establish several properties of the function $r_{\alpha,\beta}(y)$ which will be helpful later.

Lemma 2.5. *The function $r_{\alpha,\beta}(y)$ is homogeneous (i.e., $r_{\alpha,\beta}(ty) = tr_{\alpha,\beta}(y)$ for all $t \geq 0$.)*

Proof. For any $t > 0$, we have that $J_{\alpha,\beta}(r_{\alpha,\beta}(ty) + ty) \geq J_{\alpha,\beta}(x + ty)$ for all $x \in X_k$. By the homogeneity of $J_{\alpha,\beta}$, we therefore have $J_{\alpha,\beta}\left(\frac{r_{\alpha,\beta}(ty)}{t} + y\right) \geq J_{\alpha,\beta}\left(\frac{x}{t} + y\right)$ for all $x \in X_k$. But this implies that $\frac{r_{\alpha,\beta}(ty)}{t} = r_{\alpha,\beta}(y)$, and therefore $r_{\alpha,\beta}$ is homogeneous.

For $t = 0$, we need only to show $r_{\alpha,\beta}(0) = 0$. Clearly $J_{\alpha,\beta}(0) = 0$. We will show that $J_{\alpha,\beta}(x) < 0$ for all $x \in X_k \setminus \{0\}$, and therefore, $0 = \max_{x \in X_k} J_{\alpha,\beta}(x) = r_{\alpha,\beta}(0)$.

Since $\|x\|_{(c,\sigma)}^2 \leq \mu_k \|x\|_{(m,\rho)}^2$ (see [7, Corollary 2.2]), we observe that

$$\begin{aligned} J_{\alpha,\beta}(x) &= \frac{1}{2} \left(\|x\|_{(c,\sigma)}^2 - \alpha \|x^+\|_{(m,\rho)}^2 - \beta \|x^-\|_{(m,\rho)}^2 \right) \\ &\leq \frac{1}{2} \left(\mu_k \|x\|_{(m,\rho)}^2 - \alpha \|x^+\|_{(m,\rho)}^2 - \beta \|x^-\|_{(m,\rho)}^2 \right) \\ &\leq \frac{1}{2} \left(\mu_k \|x\|_{(m,\rho)}^2 - \alpha \|x^+\|_{(m,\rho)}^2 - \alpha \|x^-\|_{(m,\rho)}^2 \right) \\ &\leq \frac{1}{2} (\mu_k - \alpha) \|x\|_{(m,\rho)}^2 < 0, \end{aligned}$$

for all $x \in X_k \setminus \{0\}$. Hence, $r_{\alpha,\beta}(0) = 0$, and therefore $r_{\alpha,\beta}(ty) = tr_{\alpha,\beta}(y)$ for all $t \geq 0$ and $y \in Y$. \square

Lemma 2.6. *For each $y \neq 0$, $r_{\alpha,\beta}(y) + y$ changes sign.*

Proof. Suppose to the contrary that $u = r_{\alpha,\beta}(y) + y$ is nonnegative and strictly positive on some set of positive measure, say Ω_1 . Since $u \in H^1$, $u = v + \sum k_n \phi_n$ for $k_n = \langle u, \phi_n \rangle_{(m,\rho)}$ and some $v \in V_{(m,\rho)}$. We note that $k_1 = \langle u, \phi_1 \rangle_{(m,\rho)} > 0$ since $\phi_1 > 0$ on Ω_1 and $u \notin V_{(m,\rho)}$.

Since $\phi_1 \in X_k \forall k \geq 1$ and $r_{\alpha,\beta}(y)$ maximizes $J_{\alpha,\beta}$ on X_k , we have

$$\begin{aligned} 0 &= J'_{\alpha,\beta}(u) \cdot \phi_1 \\ &= \langle u, \phi_1 \rangle_{(c,\sigma)} - \alpha \langle u, \phi_1 \rangle_{(m,\rho)} + (\beta - \alpha) \langle u^-, \phi_1 \rangle_{(m,\rho)} \\ &= \langle u, \phi_1 \rangle_{(c,\sigma)} - \alpha \langle u, \phi_1 \rangle_{(m,\rho)} \\ &= k_1 \|\phi_1\|_{(c,\sigma)}^2 - \alpha k_1 \|\phi_1\|_{(m,\rho)}^2 \\ &= k_1 (\mu_1 - \alpha) \|\phi_1\|_{(m,\rho)}^2 < 0, \end{aligned}$$

which is a contradiction. An identical contradiction can be reached in the case that we assume u is nonpositive and strictly negative on some set of positive measure. Hence, $r_{\alpha,\beta}(y) + y$ must change sign for $y \neq 0$. \square

To be precise about the result of the following lemma, let us consider the space \tilde{Y} , which is the set of points in Y endowed with the topology generated by $\|\cdot\|_{(m,\rho)}$.

Lemma 2.7. $r_{\alpha,\beta}(y)$ is locally Lipschitz continuous as a function of $\mathbb{R}^2 \times \tilde{Y}$ into X_k .

Proof. Take $x_i = r_{\alpha_i,\beta_i}(y_i)$. By the definition of $r_{\alpha_i,\beta_i}(y_i)$, we have that

$$(J'_{\alpha_2,\beta_2}(r_{\alpha_2,\beta_2}(y_2) + y_2) - J'_{\alpha_1,\beta_1}(r_{\alpha_1,\beta_1}(y_1))) \cdot (r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)) = 0,$$

and hence by Lemma 2.3, we have that

$$\begin{aligned} & \delta \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(c,\sigma)}^2 \\ & \leq s_2 (\|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)} + \|y_2 - y_1\|_{(m,\rho)}) \|y_2 - y_1\|_{(m,\rho)} \\ & \quad + |\alpha_2 - \alpha_1| \|r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)} \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)} \\ & \quad + |s_2 - s_1| \|r_{\alpha_1,\beta_1}(y_1) + r_{\alpha_2,\beta_2}(y_2)\|_{(m,\rho)} \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)}, \end{aligned}$$

Applying a Poincaré-type inequality (see Corollary 2.2 in [7]), we obtain

$$\begin{aligned} & \delta \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(c,\sigma)}^2 \\ & \leq s_2 \left(\frac{1}{\mu_1} \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(c,\sigma)} + \|y_2 - y_1\|_{(m,\rho)} \right) \|y_2 - y_1\|_{(m,\rho)} \\ & \quad + |\alpha_2 - \alpha_1| \|r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)} \frac{1}{\mu_1} \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(c,\sigma)} \\ & \quad + |s_2 - s_1| \|r_{\alpha_1,\beta_1}(y_1) + r_{\alpha_2,\beta_2}(y_2)\|_{(m,\rho)} \frac{1}{\mu_1} \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(c,\sigma)}, \end{aligned} \tag{2.6}$$

Now, for a given y_1 , let $c_1 = \|r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)}$, $c_2 = \|r_{\alpha_1,\beta_1}(y_1) + y_1\|_{(m,\rho)}$, and $z = \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(c,\sigma)}$. It follows from (2.6) that

$$\delta z^2 \leq (\|y_2 - y_1\|_{(m,\rho)} + c_1|\alpha_2 - \alpha_1| + c_2|s_2 - s_1|) \frac{1}{\mu_1} z + \|y_2 - y_1\|_{(m,\rho)}^2$$

Taking $\gamma := (\|y_2 - y_1\|_{(m,\rho)} + c_1|\alpha_2 - \alpha_1| + c_2|s_2 - s_1|)$, we observe that $\|y_2 - y_1\|_{(m,\rho)} \leq \gamma$, and therefore,

$$\delta z^2 \leq \frac{\gamma}{\mu_1} z + \gamma^2.$$

Therefore, $z \leq C(\delta)\gamma$, and the lemma is proven. \square

Note that in the case $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$, γ is independent of c_1 and c_2 . Therefore, since $C(\delta)$ is also independent of y_1 and y_2 , we have the following corollary.

Corollary 2.8. For a given α and β , $r_{\alpha,\beta} : \tilde{Y} \rightarrow X_k$ is globally Lipschitz continuous.

Lemma 2.9. There exists a $C > 0$ such that $\|r_{\alpha,\beta}(y)\|_{(c,\sigma)} \leq C\|y\|_{(m,\rho)}$.

Proof. Suppose $y_2 = y$ and $y_1 = 0$ are fixed and further suppose that $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$. Then $x_2 = r_{\alpha_2,\beta_2}(y_2) = r_{\alpha,\beta}(y)$ and $x_1 = r_{\alpha_1,\beta_1}(y_1) = r_{\alpha,\beta}(0) = 0$. Then (2.6) reduces to

$$\delta \|r_{\alpha,\beta}(y)\|_{(c,\sigma)}^2 \leq \left(\frac{1}{\mu_1} \|r_{\alpha,\beta}(y)\|_{(c,\sigma)} + \|y\|_{(m,\rho)} \right) \|y\|_{(m,\rho)}.$$

We may solve this inequality to observe that $\delta \|r_{\alpha,\beta}(y)\|_{(c,\sigma)} \leq C(\delta)\|y\|_{(m,\rho)}$ where $C(\delta) = \frac{1}{2\mu_1\delta} + \sqrt{\frac{1}{\delta} + \frac{1}{4\mu_1^2\delta^2}} > 0$. Note that C is a decreasing function of δ , and

therefore, if $\alpha - \mu_k = \epsilon > 0$, then we can choose $\bar{\delta} = \frac{\epsilon}{\mu_k} < \frac{\alpha}{\mu_k} - 1 = \delta$ such that

$$\bar{\delta} \|r_{\alpha,\beta}(y)\|_{c,\sigma} \leq \delta \|r_{\alpha,\beta}(y)\|_{c,\sigma} \leq C(\delta) \|y\|_{(m,\rho)} \leq C(\bar{\delta}) \|y\|_{(m,\rho)}.$$

□

The function $r_{\alpha,\beta}(y)$ also satisfies a compactness condition, namely:

Lemma 2.10. *Let $\{(\alpha_n, \beta_n)\}$ be a bounded sequence in \mathbb{R}^2 satisfying $\mu_k < \alpha_n < \mu_{k+1}$ and $\alpha_n < \beta_n$ and let $\{y_n\}$ be a bounded sequence in Y . Then there exist subsequences, again called, $\{(\alpha_n, \beta_n)\}$ and $\{y_n\}$ such that $(\alpha_n, \beta_n) \rightarrow (\alpha, \beta)$ in \mathbb{R}^2 , $y_n \rightarrow y$ in Y , $y_n \rightarrow y$ in \tilde{Y} , and $r_{\alpha_n, \beta_n}(y_n) \rightarrow r_{\alpha, \beta}(y)$ in X_k .*

Proof. There exists a subsequence of $\{(\alpha_n, \beta_n)\}$ converging to (α, β) in \mathbb{R}^2 by the Bolzano-Weierstrauss Theorem, call it again $\{(\alpha_n, \beta_n)\}$. Then there exists a subsequence of $\{y_n\}$ converging weakly to y in Y by the fact that $H^1(\Omega)$ is reflexive. We again call that subsequence $\{y_n\}$. Finally, by the Rellich-Kondrachov Theorem and the compactness of the trace operator given $m \in L^\infty(\Omega)$ and $\rho \in L^\infty(\partial\Omega)$, there exists a subsequence of $\{y_n\}$ converging strongly to y in \tilde{Y} , called again $\{y_n\}$. Hence, by the continuity of $r_{\alpha,\beta}$ established in Lemma 2.7, we have $r_{\alpha_n, \beta_n}(y_n) \rightarrow r_{\alpha, \beta}(y)$ in X . □

Finally, we observe the following property of $r_{\alpha,\beta}$.

Lemma 2.11. *If $u \in H^1(\Omega)$ is a critical point of $J_{\alpha,\beta}$, then $u = r_{\alpha,\beta}(y) + y$ for some $y \in Y$.*

Proof. Since u is a critical point of $J_{\alpha,\beta}$, $J'_{\alpha,\beta}(u) \cdot v = 0$ for all $v \in H^1(\Omega)$. Since $H^1(\Omega) = X_k \oplus Y$, we may write $u = x + y$ where $x \in X_k$ and $y \in Y$. We observe that $0 = J'_{\alpha,\beta}(u) \cdot x = J'_{\alpha,\beta}(x) \cdot x$, showing that x is a critical point of $J_{\alpha,\beta}$ on the set $y + X$. But $J_{\alpha,\beta}$ is strictly concave on $y + X$ and its unique maximizer is defined as $r_{\alpha,\beta}(y)$. So $x = r_{\alpha,\beta}(y)$ and hence $u = r_{\alpha,\beta}(y) + y$. □

3. REDUCING THE FUNCTIONAL

Motivated by Lemma 2.11, we now define the restricted functional $\tilde{J}_{\alpha,\beta} : Y \rightarrow \mathbb{R}$ by $\tilde{J}_{\alpha,\beta}(y) := J_{\alpha,\beta}(r_{\alpha,\beta}(y) + y)$. We begin by establishing some properties of this new functional.

Lemma 3.1. *The functional $\tilde{J}_{\alpha,\beta} \in C^1(Y, \mathbb{R})$ and $\tilde{J}'_{\alpha,\beta}(y) = J'_{\alpha,\beta}(r_{\alpha,\beta}(y) + y)$ for all $y \in Y$.*

Proof. We will establish this claim by showing that

$$\tilde{J}_{\alpha,\beta}(y_2) - \tilde{J}_{\alpha,\beta}(y_1) = J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1) + y_1) \cdot (y_2 - y_1) + o(\|y_2 - y_1\|_{(m,\rho)}).$$

In addition to showing that $\tilde{J}_{\alpha,\beta} \in C^1(Y, \mathbb{R})$, this will also establish that $\tilde{J}'_{\alpha,\beta}(y) = J'_{\alpha,\beta}(r_{\alpha,\beta}(y) + y)$. First, note that

$$\begin{aligned} & \tilde{J}_{\alpha,\beta}(y_2) - \tilde{J}_{\alpha,\beta}(y_1) \\ &= J_{\alpha,\beta}(r_{\alpha,\beta}(y_2) + y_2) - J_{\alpha,\beta}(r_{\alpha,\beta}(y_1) + y_1) \\ &\leq J_{\alpha,\beta}(r_{\alpha,\beta}(y_2) + y_2) - J_{\alpha,\beta}(r_{\alpha,\beta}(y_2) + y_1) \\ &= J'_{\alpha,\beta}(r_{\alpha,\beta}(y_2) + y_1) \cdot (y_2 - y_1) + o(\|y_2 - y_1\|_{(m,\rho)}) \\ &= J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1) + y_1) \cdot (y_2 - y_1) + \left(J'_{\alpha,\beta}(r_{\alpha,\beta}(y_2) + y_1) \right. \\ &\quad \left. - J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1) + y_1) \right) \cdot (y_2 - y_1) + o(\|y_2 - y_1\|_{(m,\rho)}) \end{aligned} \tag{3.1}$$

by the maximizing property of $r_{\alpha,\beta}$, the Lipschitz continuity of $r_{\alpha,\beta}$, and the differentiability of $J_{\alpha,\beta}$. By the continuity of $r_{\alpha,\beta}$ and $J'_{\alpha,\beta}$, we note that

$$\left(J'_{\alpha,\beta}(r_{\alpha,\beta}(y_2) + y_1) - J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1) + y_1) \right) \cdot (y_2 - y_1) = o(\|y_2 - y_1\|_{(m,\rho)}),$$

and hence (3.1) reduces to

$$\tilde{J}_{\alpha,\beta}(y_2) - \tilde{J}_{\alpha,\beta}(y_1) \leq J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1) + y_1) \cdot (y_2 - y_1) + o(\|y_2 - y_1\|_{(m,\rho)}).$$

A similar argument will show that

$$\tilde{J}_{\alpha,\beta}(y_2) - \tilde{J}_{\alpha,\beta}(y_1) \geq J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1) + y_1) \cdot (y_2 - y_1) + o(\|y_2 - y_1\|_{(m,\rho)}),$$

and hence the claim is proven. □

Remark 3.2. If we knew $r_{\alpha,\beta}$ to be differentiable, this result would be a simple consequence of the chain rule. However, in general, this is not the case.

Given that we have now established that $\tilde{J}_{\alpha,\beta} \in C^1(Y, \mathbb{R})$, we may improve upon Lemma 2.11.

Lemma 3.3. *The element $y \in Y$ is a critical point of $\tilde{J}_{\alpha,\beta}$ if and only if $r_{\alpha,\beta}(y) + y$ is a critical point of $J_{\alpha,\beta}$.*

Proof. First, assume that $r_{\alpha,\beta}(y) + y$ is a critical point of $J_{\alpha,\beta}$. Then $J'_{\alpha,\beta}(r_{\alpha,\beta}(y) + y) \cdot v = 0$ for all $v \in H^1(\Omega)$ (and in particular, for all $v \in Y$). By Lemma 3.1, this implies that $\tilde{J}'_{\alpha,\beta}(y) \cdot v = 0$ for all $v \in Y$, and y is a critical point of $\tilde{J}_{\alpha,\beta}$.

Now, assume that y is a critical point of $\tilde{J}_{\alpha,\beta}$. As before, we then have that $J'_{\alpha,\beta}(r_{\alpha,\beta}(y) + y) \cdot v = 0$ for all $v \in Y$. However, since $r_{\alpha,\beta}(y)$ maximizes $J_{\alpha,\beta}(x + y)$ for all $x \in X_k$, we also have that $J'_{\alpha,\beta}(r_{\alpha,\beta}(y) + y) \cdot x = 0$ for all $x \in X_k$. Hence, since $H^1(\Omega) = X_k \oplus Y$, we have $J'_{\alpha,\beta}(r_{\alpha,\beta}(y) + y) \cdot w = 0$ for all $w \in H^1(\Omega)$. □

Now, we observe a homogeneity property of $\tilde{J}_{\alpha,\beta}$.

Lemma 3.4. *The functional $\tilde{J}_{\alpha,\beta}(ty) = t^2 \tilde{J}_{\alpha,\beta}(y)$ for all $t \geq 0$ and $y \in Y$.*

The result follows immediately from the homogeneity of $J_{\alpha,\beta}$ and the homogeneity of $r_{\alpha,\beta}$ from Lemma 2.5. An important consequence of this lemma easily follows.

Lemma 3.5. *If $y \in Y$ is a critical point of $\tilde{J}_{\alpha,\beta}$, then $\tilde{J}_{\alpha,\beta}(y) = 0$.*

Proof. Differentiating the identity $\tilde{J}_{\alpha,\beta}(ty) = t^2 \tilde{J}_{\alpha,\beta}(y)$ with respect to t , we find that $\tilde{J}'_{\alpha,\beta}(ty) \cdot y = 2t \tilde{J}_{\alpha,\beta}(y)$. Setting $t = 1$, the result immediately follows. □

As with $J_{\alpha,\beta}$, it will occasionally be helpful to think of $\tilde{J}_{\alpha,\beta}$ as a function on $\mathbb{R}^2 \times Y$, which we denote $\tilde{J}(\alpha, \beta, y)$.

Lemma 3.6. *For each fixed $y \neq 0$, the functional $\tilde{J}(\alpha, \beta, y) := \tilde{J}_{\alpha,\beta}(y)$ is strictly decreasing in α and β .*

Proof. Assume that $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$, with at least one of these inequalities strict. Then

$$\begin{aligned} \tilde{J}(\alpha_2, \beta_2, y) &= J(\alpha_2, \beta_2, r(\alpha_2, \beta_2, y) + y) \\ &= \frac{1}{2} [\|r(\alpha_2, \beta_2, y) + y\|_{(c,\sigma)}^2 - \alpha_2 \| (r(\alpha_2, \beta_2, y) + y)^+ \|_{(m,\rho)}^2 \\ &\quad - \beta_2 \| (r(\alpha_2, \beta_2, y) + y)^- \|_{(m,\rho)}^2]. \end{aligned} \quad (3.2)$$

Since $r(\alpha_2, \beta_2, y) + y$ is sign-changing for $y \neq 0$ by Lemma 2.6, it follows that

$$\| (r(\alpha_2, \beta_2, y) + y)^+ \|_{(m,\rho)} \| (r(\alpha_2, \beta_2, y) + y)^- \|_{(m,\rho)} > 0,$$

and hence, since at least one of the inequalities $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$ is strict, we have from (3.2) that

$$\begin{aligned} \tilde{J}(\alpha_2, \beta_2, y) &< \frac{1}{2} [\|r(\alpha_2, \beta_2, y) + y\|_{(c,\sigma)}^2 - \alpha_1 \| (r(\alpha_2, \beta_2, y) + y)^+ \|_{(m,\rho)}^2 \\ &\quad - \beta_1 \| (r(\alpha_2, \beta_2, y) + y)^- \|_{(m,\rho)}^2] \\ &= J(\alpha_1, \beta_1, r(\alpha_2, \beta_2, y) + y). \end{aligned} \quad (3.3)$$

But recalling the maximizing property of $r_{\alpha,\beta}$ (see (2.5)), we must have that

$$J(\alpha_1, \beta_1, r(\alpha_2, \beta_2, y) + y) \leq J(\alpha_1, \beta_1, r(\alpha_1, \beta_1, y) + y) = \tilde{J}(\alpha_1, \beta_1, y). \quad (3.4)$$

Combining (3.3) and (3.4) gives the desired result, that $\tilde{J}(\alpha_2, \beta_2, y) < \tilde{J}(\alpha_1, \beta_1, y)$ for each $y \neq 0$. \square

Lemma 3.7. *Given any $K > 0$, there exists $C > 0$ such that*

$$|\tilde{J}(\alpha_2, \beta_2, x) - \tilde{J}(\alpha_1, \beta_1, x)| \leq C (|\alpha_2 - \alpha_1| + |\beta_2 - \beta_1|)$$

on $R(K) := \{(\alpha_1, \alpha_2, \beta_1, \beta_2, y) \in \mathbb{R}^4 \times H^1(\Omega) : \max\{|\alpha_1|, |\alpha_2|, |\beta_1|, |\beta_2|, \|y\|_{(c,\sigma)}\} \leq K\}$.

Proof. First, we establish that the functional J is uniformly Lipschitz in α, β , and x . Note that

$$\begin{aligned} |J(\alpha_2, \beta_2, x) - J(\alpha_1, \beta_1, x)| &= \frac{1}{2} \left| (\alpha_2 - \alpha_1) \|x^+\|_{(m,\rho)}^2 + (\beta_2 - \beta_1) \|x^-\|_{(m,\rho)}^2 \right| \\ &\leq \frac{1}{2\mu_1} \|x\|_{(c,\sigma)}^2 (|\alpha_2 - \alpha_1| + |\beta_2 - \beta_1|) \\ &\leq \frac{1}{2\mu_1} K (|\alpha_2 - \alpha_1| + |\beta_2 - \beta_1|). \end{aligned}$$

Hence J is uniformly Lipschitz in α, β on $R(K)$. Since $J \in C^1(H^1(\Omega); \mathbb{R})$, it is also uniformly Lipschitz in x on $R(K)$

Recall from Lemma 2.7 that $r_{\alpha,\beta}$ is locally Lipschitz in α, β . Therefore, we have that $r_{\alpha,\beta}$ is uniformly Lipschitz in α, β on $R(K)$. Therefore, $\tilde{J}(\alpha, \beta, x) = J(\alpha, \beta, r(\alpha, \beta, x) + x)$ is a composition of uniformly Lipschitz functions, and hence the claim follows. \square

3.1. Minimizing in Y . By Lemma 3.3, we know that searching for critical points of $J_{\alpha,\beta}$ on H is equivalent to searching for critical points of $\tilde{J}_{\alpha,\beta}$ on Y . Further, since $\tilde{J}_{\alpha,\beta}$ is homogeneous, it is sufficient to search for critical points on the (m, ρ) -unit sphere in Y , namely $S_Y := \{y \in Y : \|y\|_{(m,\rho)} = 1\}$.

Since we assume $m, \rho \in L^\infty(\Omega)$, S_Y is weakly closed in $H^1(\Omega)$; that is, for any sequence $\{y_n\} \subset S_Y$ with $y_n \rightharpoonup y$ in $H^1(\Omega)$, we have $y_n \rightarrow y$ in Y and $y \in S_Y$. First we note several properties of $\tilde{J}_{\alpha,\beta}$ when restricted to S_Y .

Lemma 3.8. $\tilde{J}_{\alpha,\beta}$ attains a global minimum on S_Y .

Proof. First, note that

$$\begin{aligned} 2\tilde{J}_{\alpha,\beta}(y) &= 2J_{\alpha,\beta}(r_{\alpha,\beta}(y) + y) \\ &= \|r_{\alpha,\beta}(y) + y\|_{(c,\sigma)}^2 - \alpha\|(r_{\alpha,\beta}(y) + y)^+\|_{(m,\rho)}^2 - \beta\|(r_{\alpha,\beta}(y) + y)^-\|_{(m,\rho)}^2 \\ &> -\alpha\|r_{\alpha,\beta}(y) + y\|_{(m,\rho)}^2 - \beta\|r_{\alpha,\beta}(y) + y\|_{(m,\rho)}^2 \\ &> -2\beta\|r_{\alpha,\beta}(y) + y\|_{(m,\rho)}^2. \end{aligned}$$

Since $r_{\alpha,\beta}(y) \in X$ and $y \in Y$, $\langle r_{\alpha,\beta}(y), y \rangle_{(m,\rho)} = 0$ and hence

$$\begin{aligned} 2\tilde{J}_{\alpha,\beta}(y) &> -2\beta\|r_{\alpha,\beta}(y) + y\|_{(m,\rho)}^2 \\ &= -2\beta(\|r(y)\|_{(m,\rho)}^2 + \|y\|_{(m,\rho)}^2) \\ &> -2\beta\left(\frac{1}{\mu_1}\|r(y)\|_{(c,\sigma)}^2 + \|y\|_{(m,\rho)}^2\right) \\ &> -2\beta\left(\frac{C^2}{\mu_1}\|y\|_{(m,\rho)}^2 + \|y\|_{(m,\rho)}^2\right) \\ &\geq -k(\|y\|_{(m,\rho)}^2) \end{aligned}$$

where $k = 2\beta\left(\frac{C^2}{\mu_1} + 1\right)$ by Corollary 2.2 (a) in [7] and Lemma 2.9.

Now, take $M = \inf_{S_Y} \tilde{J}_{\alpha,\beta}(y) > -\infty$ (since $\|y\|_{(m,\rho)} = 1$ on S_Y) and choose $\{y_n\} \subset S_Y$ to be a minimizing sequence, that is, $\tilde{J}_{\alpha,\beta}(y_n) \rightarrow M$. So $\tilde{J}_{\alpha,\beta}(y_n)$ is bounded. We wish to show that $\|y_n\|_{(c,\sigma)}$ is also bounded. Note first that since $\tilde{J}_{\alpha,\beta}(y_n)$ is bounded and

$$\begin{aligned} 2\tilde{J}_{\alpha,\beta}(y_n) &= 2J_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n) \\ &= \|r_{\alpha,\beta}(y_n) + y_n\|_{(c,\sigma)}^2 - \alpha\|(r_{\alpha,\beta}(y_n) + y_n)^+\|_{(m,\rho)}^2 \\ &\quad - \beta\|(r_{\alpha,\beta}(y_n) + y_n)^-\|_{(m,\rho)}^2 \tag{3.5} \\ &= \|r_{\alpha,\beta}(y_n)\|_{(c,\sigma)}^2 + \|y_n\|_{(c,\sigma)}^2 - \alpha\|(r_{\alpha,\beta}(y_n) + y_n)^+\|_{(m,\rho)}^2 \\ &\quad - \beta\|(r_{\alpha,\beta}(y_n) + y_n)^-\|_{(m,\rho)}^2. \end{aligned}$$

We wish to show that all terms other than $\|y_n\|_{(c,\sigma)}^2$ in (3.5) are bounded, and hence $\|y_n\|_{(c,\sigma)}$ must also be bounded.

We recall that $\|r_{\alpha,\beta}(y_n)\|_{(c,\sigma)}^2 < C^2\|y_n\|_{(m,\rho)}^2 = C^2$ by Lemma 2.9 and by the fact that $\{y_n\} \subset S_Y$. We also note that

$$\begin{aligned} \|(r_{\alpha,\beta}(y_n) + y_n)^+\|_{(m,\rho)}^2 &\leq \|r_{\alpha,\beta}(y_n) + y_n\|_{(m,\rho)}^2 \\ &\leq \|r_{\alpha,\beta}(y_n)\|_{(m,\rho)}^2 + \|y_n\|_{(m,\rho)}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\mu_1} \|r_{\alpha,\beta}(y_n)\|_{(c,\sigma)}^2 + \|y_n\|_{(m,\rho)}^2 \\
&\leq \frac{C^2}{\mu_1} \|y_n\|_{(m,\rho)}^2 + \|y_n\|_{(m,\rho)}^2 \\
&= \frac{C^2}{\mu_1} + 1,
\end{aligned}$$

by [7, Corollary 2.2(a)], Lemma 2.9, and the fact that $\{y_n\} \subset S_Y$. An identical argument will show that $\|(r_{\alpha,\beta}(y_n) + y_n)^-\|_{(m,\rho)}^2 \leq \frac{C^2}{\mu_1} + 1$. Hence, we have shown via equation (3.5) that $\|y_n\|_{(c,\sigma)}$ is also bounded.

Hence, by Lemma 2.10, we may choose a subsequence, call it again $\{y_n\}$, with $y_n \xrightarrow{(c,\sigma)} y_0$, $y_n \xrightarrow{(m,\rho)} y_0$ with $\|y_0\|_{(m,\rho)} = 1$, and $r_{\alpha,\beta}(y_n) \xrightarrow{(c,\sigma)} r_{\alpha,\beta}(y_0)$. So taking the limit inferior of both sides of (3.5) as $n \rightarrow \infty$, we see that

$$\begin{aligned}
2M &= \liminf_{n \rightarrow \infty} 2\tilde{J}_{\alpha,\beta}(y_n) \\
&= \|r_{\alpha,\beta}(y_0)\|_{(c,\sigma)}^2 + \liminf_{n \rightarrow \infty} \|y_n\|_{(c,\sigma)}^2 \\
&\quad - \alpha \|(r_{\alpha,\beta}(y_0) + y_0)^+\|_{(m,\rho)}^2 - \beta \|(r_{\alpha,\beta}(y_0) + y_0)^-\|_{(m,\rho)}^2 \\
&\geq \|r_{\alpha,\beta}(y_0)\|_{(c,\sigma)}^2 + \|y_0\|_{(c,\sigma)}^2 - \alpha \|(r_{\alpha,\beta}(y_0) + y_0)^+\|_{(m,\rho)}^2 \\
&\quad - \beta \|(r_{\alpha,\beta}(y_0) + y_0)^-\|_{(m,\rho)}^2 \\
&= 2\tilde{J}_{\alpha,\beta}(y_0),
\end{aligned}$$

by the weak lower semicontinuity of the (c, σ) norm. But then $M \geq \tilde{J}_{\alpha,\beta}(y_0)$ with $y_0 \in S_Y$ and hence we must have $\tilde{J}_{\alpha,\beta}(y_0) = M$ as desired. \square

Lemma 3.9. y_0 is a nontrivial critical point of $\tilde{J}_{\alpha,\beta}$ if and only if $\frac{y_0}{\|y_0\|_{(m,\rho)}}$ is a critical point of $\tilde{J}_{\alpha,\beta}$ restricted to S_Y and $\tilde{J}_{\alpha,\beta}(y_0) = 0$.

Proof. If y_0 is a nontrivial critical point of $\tilde{J}_{\alpha,\beta}$, then by Lemma 3.5, $\tilde{J}_{\alpha,\beta}(y_0) = 0$. Furthermore, since y_0 is a critical point of $\tilde{J}_{\alpha,\beta}$, we may differentiate both sides of the equation in Lemma 3.4 with respect to y and set $t = 1/\|y_0\|_{(m,\rho)}$ to see that

$$0 = \frac{1}{\|y_0\|_{(m,\rho)}} \tilde{J}'_{\alpha,\beta}(y_0) \cdot y = \tilde{J}'_{\alpha,\beta}\left(\frac{y_0}{\|y_0\|_{(m,\rho)}}\right) \cdot y$$

holds for all $y \in Y$. So in particular, it holds for $y \in S_Y$ and the forward direction is established.

Now, let $y_0/\|y_0\|_{(m,\rho)}$ be a critical point of $\tilde{J}_{\alpha,\beta}$ restricted to S_Y and let $\tilde{J}_{\alpha,\beta}(y_0) = 0$. Then as in the previous case, we have

$$0 = \tilde{J}'_{\alpha,\beta}\left(\frac{y_0}{\|y_0\|_{(m,\rho)}}\right) \cdot y = \frac{1}{\|y_0\|_{(m,\rho)}} \tilde{J}'_{\alpha,\beta}(y_0) \cdot y$$

for all $y \in S_Y$. But note that, for any $\hat{y} \in Y$, we may write $\hat{y} = ty$ for some $y \in S_Y$. So,

$$\tilde{J}'_{\alpha,\beta}(y_0) \cdot \hat{y} = \tilde{J}'_{\alpha,\beta}(y_0) \cdot (ty) = t\tilde{J}'_{\alpha,\beta}(y_0) \cdot y = 0.$$

So y_0 is a critical point of $\tilde{J}_{\alpha,\beta}$ as desired. \square

Lemma 3.10. A function $u \in H^1(\Omega)$ is a nontrivial critical point of $J_{\alpha,\beta}$ if and only if $u = r_{\alpha,\beta}(y_0) + y_0$ where $y_0/\|y_0\|_{(m,\rho)}$ is a critical point of $\tilde{J}_{\alpha,\beta}$ restricted to S_Y and $\tilde{J}_{\alpha,\beta}(y_0) = 0$.

The above lemma follows from combining Lemma 3.3 and Lemma 3.9. We now define $M(\alpha, \beta) = \min_{y \in S_Y} J_{\alpha, \beta}(y)$.

Lemma 3.11. *The function $M(\alpha, \beta)$ is Lipschitz continuous and is strictly decreasing as a function of both α and β . Moreover, $M(\alpha, \alpha) > 0$.*

Proof. Let (α_1, β_1) and (α_2, β_2) be points in the plane, and let y_1 and y_2 be the corresponding minimizers on S_Y (i.e. $M(\alpha_k, \beta_k) = \tilde{J}_{\alpha_k, \beta_k}(y_k)$ for $k = 1, 2$). Let $u_{ij} = r_{\alpha_i, \beta_i}(y_j) + y_j$ for $i, j = 1, 2$. Then

$$\begin{aligned}
 M(\alpha_i, \beta_i) &= \tilde{J}_{\alpha_i, \beta_i}(y_i) \\
 &\leq \tilde{J}_{\alpha_i, \beta_i}(y_j) \\
 &= J_{\alpha_i, \beta_i}(r_{\alpha_i, \beta_i}(y_j) + y_j) \\
 &= J_{\alpha_j, \beta_j}(r_{\alpha_i, \beta_i}(y_j) + y_j) + \frac{1}{2}(\alpha_j - \alpha_i)\|u_{ij}^+\|_{(m, \rho)}^2 + \frac{1}{2}(\beta_j - \beta_i)\|u_{ij}^-\|_{(m, \rho)}^2 \\
 &\leq J_{\alpha_j, \beta_j}(r_{\alpha_j, \beta_j}(y_j) + y_j) + \frac{1}{2}(\alpha_j - \alpha_i)\|u_{ij}^+\|_{(m, \rho)}^2 + \frac{1}{2}(\beta_j - \beta_i)\|u_{ij}^-\|_{(m, \rho)}^2 \\
 &= M(\alpha_j, \beta_j) + \frac{1}{2}(\alpha_j - \alpha_i)\|u_{ij}^+\|_{(m, \rho)}^2 + \frac{1}{2}(\beta_j - \beta_i)\|u_{ij}^-\|_{(m, \rho)}^2
 \end{aligned} \tag{3.6}$$

by the minimizing property of y_i and the maximizing property of r_{α_j, β_j} . This inequality holds in the case $i = 1$ and $j = 2$, as well as the case $i = 2$ and $j = 1$. Hence

$$|M(\alpha_2, \beta_2) - M(\alpha_1, \beta_1)| \leq c(|\alpha_2 - \alpha_1| + |\beta_2 - \beta_1|)$$

where $c = \frac{1}{2} \max\{\|u_{12}\|_{(m, \rho)}^2, \|u_{21}\|_{(m, \rho)}^2\}$. Note that if $\alpha_2 \geq \alpha_1$ and $\beta_2 \geq \beta_1$ with at least one of the inequalities strict, then $M(\alpha_2, \beta_2) < M(\alpha_1, \beta_1)$ by taking $i = 2$ and $j = 1$ in (3.6). This follows from the fact that u_{ij} must be sign-changing by Lemma 2.6.

In the case $\alpha = \beta$, for every $w \in H^1(\Omega)$, we may write $w = u + v$ where $u \in H_{(m, \rho)}^1$ and $v \in V_{(m, \rho)}$ have

$$\begin{aligned}
 2J_{\alpha, \beta}(w) &= 2J_{\alpha, \alpha}(w) \\
 &= \|u + v\|_{(c, \sigma)}^2 - \alpha\|u + v\|_{(m, \rho)}^2 \\
 &= \|u\|_{(c, \sigma)}^2 + \|v\|_{(c, \sigma)}^2 - \alpha\|u\|_{(m, \rho)}^2 + \|v\|_{(m, \rho)}^2 \\
 &= \|u\|_{(c, \sigma)}^2 - \alpha\|u\|_{(m, \rho)}^2 + \|v\|_{(c, \sigma)}^2 \\
 &= \sum_{i=1}^{\infty} (\mu_i - \alpha)|c_i|^2 + \|v\|_{(c, \sigma)}^2
 \end{aligned}$$

by [7, Theorem 2.1(iii)] where

$$c_i = \frac{1}{\mu_i} \langle u, \phi_i \rangle_{(c, \sigma)} = \langle u, \phi_i \rangle_{(m, \rho)}.$$

Since $\mu_k < \alpha < \mu_{k+1}$, we note that the coefficients $(\mu_i - \alpha)$ are negative for $i \leq k$, positive for $i \geq k + 1$, and are increasing in i . Writing $u = x + y$ where $x \in X_k$ and $y \in Y_k$, we note that, if we maximize in the X_k direction, the maximum occurs when $c_i = 0$ for all $i \leq k$ since $(\mu_i - \alpha)$ are negative for $i \leq k$. In other words,

$r_{\alpha,\alpha}(y) \equiv 0$. Therefore,

$$2\tilde{J}_{\alpha,\alpha}(y) = 2J_{\alpha,\alpha}(y) = \sum_{i=k+1}^{\infty} (\mu_i - \alpha)|c_i|^2 \text{ for all } y \in Y_k. \quad (3.7)$$

We now wish to show that $M(\alpha, \alpha) = \inf_{y \in S_Y} \tilde{J}_{\alpha,\alpha}(y) > 0$. Taking

$$f(c_{k+1}, c_{k+2}, \dots) = \sum_{i=k+1}^{\infty} (\mu_i - \alpha)|c_i|^2, \quad g(c_{k+1}, c_{k+2}, \dots) = \sum_{i=k+1}^{\infty} |c_i|^2,$$

we apply the method of Lagrange multipliers to find the critical points of f subject to the constraint $g(c_{k+1}, c_{k+2}, \dots) = 1$. Setting $\nabla f = \lambda \nabla g$, we obtain $2(\mu_i - \alpha)c_i = 2\lambda c_i$ for $i \geq k+1$. Hence, critical points occur when $c_j = \pm 1$ for some $j \geq k+1$ and $c_i = 0$ for all $i \neq j$ (corresponding to the Lagrange multiplier $\lambda = \mu_j - \alpha$). Since the coefficients $(\mu_i - \alpha)$ are positive for $i \geq k+1$ and increasing, the minimizing choice occurs when $c_{k+1} = \pm 1$ and $c_i = 0$ for all $i > k+1$. Hence, the minimizer is $y = \pm \phi_{k+1}$ and $M(\alpha, \alpha) = \tilde{J}_{\alpha,\alpha}(\phi_{k+1}) + \|v\|_{(c,\sigma)}^2 = \frac{1}{2}(\mu_{k+1} - \alpha) + \|v\|_{(c,\sigma)}^2 > 0$. \square

Not only $M(\alpha, \alpha) > 0$, we can also make an additional estimate which will later help in establishing bounds for the Fučík spectrum.

Lemma 3.12. $M(\alpha, \mu_{k+1}) > 0$.

Proof. Let $y \in S_Y$ and let $y = z + v$ where $z \in Y_k$ and $v \in V_{(m,\rho)}$. Then, by the maximizing property

$$\begin{aligned} \tilde{J}_{\alpha,\mu_{k+1}} &= J_{\alpha,\mu_{k+1}}(r_{\alpha,\mu_{k+1}}(y) + y) \\ &\geq J_{\alpha,\mu_{k+1}}(y) \\ &= \frac{1}{2} \left(\|y\|_{(c,\sigma)}^2 - \alpha \|y^+\|_{(m,\rho)}^2 - \mu_{k+1} \|y^-\|_{(m,\rho)}^2 \right) \\ &\geq \frac{1}{2} \left(\|y\|_{(c,\sigma)}^2 - \mu_{k+1} \|y\|_{(m,\rho)}^2 \right) \\ &= \frac{1}{2} \left(\|z\|_{(c,\sigma)}^2 - \mu_{k+1} \|z\|_{(m,\rho)}^2 + \|v\|_{(c,\sigma)}^2 \right) \\ &= \frac{1}{2} \sum_{i=k+1}^{\infty} (\mu_i - \mu_{k+1})|c_i|^2 + \|v\|_{(c,\sigma)}^2 \geq 0. \end{aligned}$$

We note that, of the last two inequalities above, at least one must be strict. If the last inequality is in fact an equality, then $c_{k+1} = 1$ and $c_i = 0$ for all $i > k+1$. But this would imply that $y = \pm \phi_{k+1}$, in which case y^+ is nontrivial, and the previous inequality was strict. So $M(\alpha, \mu_{k+1}) > 0$. \square

3.2. Variational characterization of the Fučík spectrum. All of the previous lemmas lead to the following theorem.

Theorem 3.13. *Let $\mu_k < \alpha < \mu_{k+1}$. Then one of the following is true:*

- (1) $M(\alpha, \beta) > 0$ for all $\beta \geq \alpha$, which implies that $(\alpha, \beta) \notin \Sigma$.
- (2) There is a unique $\beta(\alpha) > \mu_{k+1}$ such that $M(\alpha, \beta(\alpha)) = 0$, which implies that $(\alpha, \beta(\alpha)) \in \Sigma$ but $(\alpha, \beta) \notin \Sigma$ for all $\alpha \leq \beta < \beta(\alpha)$.

Lemma 3.14. *The curve $(\alpha, \beta(\alpha))$ is Lipschitz continuous, strictly decreasing, and contains the point (μ_{k+1}, μ_{k+1}) .*

Proof. Consider two points $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \Sigma$ with $\alpha_2 > \alpha_1$. Let $y_i \in S_Y$ be a minimizer of $\tilde{J}_{\alpha_i, \beta_i}(y)$. Then $\tilde{J}_{\alpha_i, \beta_i}(y_i) = 0$ and $\tilde{J}_{\alpha_i, \beta_i}(y) \geq 0$ for all $y \in S_Y$, and therefore by the homogeneity of $\tilde{J}_{\alpha_i, \beta_i}$, $\tilde{J}_{\alpha_i, \beta_i}(y) \geq 0$ for all $y \in Y$. Let $u_i = r_{\alpha_i, \beta_i}(y_i) + y_i$. Since $(\alpha_1, \beta_1) \in \Sigma$, we have

$$\begin{aligned} 0 &= M(\alpha_1, \beta_1) \\ &= 2\tilde{J}_{\alpha_1, \beta_1}(y_1) \\ &= 2J_{\alpha_1, \beta_1}(u_1) \\ &= \|u_1\|_{(c, \sigma)}^2 - \alpha_1 \|u_1^+\|_{(m, \rho)}^2 - \beta_1 \|u_1^-\|_{(m, \rho)}^2 \\ &> \|u_1\|_{(c, \sigma)}^2 - \alpha_2 \|u_1^+\|_{(m, \rho)}^2 - \beta_1 \|u_1^-\|_{(m, \rho)}^2 \\ &= 2J_{\alpha_2, \beta_1}(r_{\alpha_1, \beta_1}(y_1)) \\ &\leq M(\alpha_2, \beta_1) \end{aligned}$$

where we note that the last inequality is strict since $\alpha_2 > \alpha_1$ and u_1^- is nontrivial by Lemma 2.6. Since $M(\alpha, \beta)$ is strictly decreasing in β by Lemma 3.11 and $M(\alpha_2, \beta_2) = 0$, we must have $\beta_2 < \beta_1$, which shows that $\beta(\alpha)$ is strictly decreasing as desired.

Now, we consider

$$M(\alpha_2, \beta_1) \leq J_{\alpha_2, \beta_1}(u_2) = J_{\alpha_2, \beta_1}(u_2) - J_{\alpha_2, \beta_2}(u_2) = \frac{1}{2}(\beta_2 - \beta_1)\|u_2^-\|_{(m, \rho)}^2,$$

since $J_{\alpha_2, \beta_2}(u_2) = 0$. Hence $M(\alpha_2, \beta_1) \leq \frac{1}{2}(\beta_2 - \beta_1)\|u_2^-\|_{(m, \rho)}^2 < 0$ since $\beta_1 > \beta_2$. Thus, we may rearrange the inequality to observe that

$$\begin{aligned} |\beta_2 - \beta_1| &= \beta_1 - \beta_2 \\ &\leq 2 \frac{1}{\|u_2^-\|_{(m, \rho)}^2} (-M(\alpha_2, \beta_1)) \\ &= 2 \frac{1}{\|u_2^-\|_{(m, \rho)}^2} |M(\alpha_2, \beta_1)| \\ &= 2 \frac{1}{\|u_2^-\|_{(m, \rho)}^2} |M(\alpha_2, \beta_1) - M(\alpha_1, \beta_1)| \\ &\leq 2c \frac{1}{\|u_2^-\|_{(m, \rho)}^2} |\alpha_2 - \alpha_1|, \end{aligned}$$

by the fact that $M(\alpha_1, \beta_1) = 0$ and the Lipschitz estimate for $M(\alpha, \beta)$ from Lemma 3.11. Hence, $\beta(\alpha)$ is Lipschitz continuous as desired. \square

4. NONRESONANCE PROBLEM

We are interested in the existence of weak solutions of (1.2) where $(\alpha, \beta) \in \mathbb{R}^2$ is such that $\mu_k < \alpha < \mu_{k+1}$ and $\alpha \leq \beta < \beta(\alpha)$. Since we consider a fixed k in this section, we set $X = X_k$ for notational convenience. By the characterization of the Fučík Spectrum, we know that $(\alpha, \beta) \notin \Sigma$. All properties of f, g, m, ρ, c , and σ are as outlined in Section 1.

Consider the functional associated with (1.2) defined by $I_{\alpha, \beta} := I : H^1(\Omega) \rightarrow \mathbb{R}$; with

$$I(u) = J_{\alpha, \beta}(u) - \left[\int F(x, u) + \oint G(x, u) \right], \quad (4.1)$$

where \int denotes the (volume) integral on Ω , \oint denotes the (surface) integral on $\partial\Omega$, and $F(x, u) = \int_0^u m(x)\tilde{f}(\xi)d\xi$, $G(x, u) = \int_0^u \rho(x)\tilde{g}(\xi)d\xi$, and $J_{\alpha,\beta}(u)$ is defined in (2.2). Then

$$I'(u) \cdot v = J'_{\alpha,\beta}(u)v - \left[\int f(x, u)v + \oint g(x, u)v \right] \quad \text{for all } v \in H^1(\Omega).$$

So, a critical point of I is a weak solution of (1.2).

Theorem 4.1. *Assume that $\mu_k < \alpha < \mu_{k+1}$, $\alpha \leq \beta < \beta(\alpha)$, the nonlinearities f and g are bounded continuous functions, and $m \in L^\infty(\Omega)$ and $\rho \in L^\infty(\partial\Omega)$, then problem (1.2) has at least one weak solution.*

We will use a variational argument to prove Theorem 4.1. To do so, we first prove some lemmas which will be needed in the sequel. The first lemma shows the last two terms in I have at most linear growth.

Lemma 4.2. *There is a positive constant κ such that*

$$\left| \int F(x, u) + \oint G(x, u) \right| \leq \kappa \|u\|_{(m,\rho)} \quad \text{for all } u \in H^1(\Omega), \quad (4.2)$$

where κ is independent of u .

Proof. Since \tilde{f} and \tilde{g} are bounded then there exist constant C_1 and C_2 such that $|\tilde{f}(u)| \leq C_1$ and $|\tilde{g}(u)| \leq C_2$. Therefore, $|F(x, u)| \leq C_1 m(x)|u|$ and $|G(x, u)| \leq C_2 \rho(x)|u|$. Using these estimates, Hölder inequality, and the fact that m is bounded, we obtain that

$$\left| \int F(x, u) \right| \leq \int C_1 m(x)|u| \leq \tilde{C}_1 \left(\int (m(x)|u|)^2 \right)^{1/2} \leq \kappa_1 \int m(x)u^2 \leq \kappa_1 \|u\|_{(m,\rho)},$$

where $\kappa_1 = \tilde{C}_1 \|m\|_\infty$. Similarly, $\left| \oint G(x, u) \right| \leq \kappa_2 \|u\|_{(m,\rho)}$. Thus,

$$\left| \int F(x, u) + \oint G(x, u) \right| \leq \kappa \|u\|_{(m,\rho)} \quad \text{for all } u \in H^1(\Omega). \quad \square$$

The next lemma shows the geometry of I .

Lemma 4.3. *The functional I is such that*

- (1) $I(u) \rightarrow -\infty$ as $\|u\|_{(c,\sigma)} \rightarrow \infty$, for $u \in X$; that is, I is anti-coercive on X .
- (2) I is bounded below when restricted to \mathcal{Y} , where $\mathcal{Y} := \{r_{\alpha,\beta}(y) + y : y \in Y\}$.

Proof. We shall first prove that I is anti-coercive when restricted to X . Using the fact that $\|x\|_{(c,\sigma)}^2 \leq \mu_k \|x\|_{(m,\rho)}^2$ for all $x \in X$, and $\alpha \leq \beta < \beta(\alpha)$, it follows that

$$\begin{aligned} J_{\alpha,\beta}(x) &= \frac{1}{2} [\|x\|_{(c,\sigma)}^2 - \alpha \|x^+\|_{(m,\rho)}^2 - \beta \|x^-\|_{(m,\rho)}^2] \\ &\leq \frac{1}{2} [\|x\|_{(c,\sigma)}^2 - \alpha \|x\|_{(m,\rho)}^2] \\ &\leq \frac{1}{2} [\|x\|_{(c,\sigma)}^2 - \frac{\alpha}{\mu_k} \|x\|_{(c,\sigma)}^2] \\ &= \frac{1}{2} \left(1 - \frac{\alpha}{\mu_k}\right) \|x\|_{(c,\sigma)}^2 \end{aligned}$$

Then using Lemma 4.2, we obtain

$$I(x) \leq -\eta \|x\|_{(c,\sigma)}^2 + \kappa \|x\|_{(m,\rho)} + C,$$

where $\eta = \frac{1}{2}(\frac{\alpha}{\lambda_k} - 1) > 0$. Since $\mu_1 \|x\|_{(m,\rho)} \leq \|x\|_{(c,\sigma)}$ (see [7, Corrolary 2.18]), then $I(x) \leq -\eta \|x\|_{(c,\sigma)}^2 + \frac{\kappa}{\mu_1} \|x\|_{(c,\sigma)} + C$. So, $I(u) \rightarrow -\infty$ as $\|u\|_{(c,\sigma)} \rightarrow \infty$ for $u \in X$. Thus I is anti-coercive on X .

Now, we shall prove that I is bounded below when restricted to \mathcal{Y} . By the assumption $\beta < \beta(\alpha)$ and Theorem 3.13, it follows that $\min_{y \in S_Y} \tilde{J}_{\alpha,\beta}(y) = M(\alpha, \beta)$ and $M(\alpha, \beta) > 0$. Then for $y \neq 0$ and $y \in Y$, we have that

$$J_{\alpha,\beta}(r_{\alpha,\beta}(y) + y) = \tilde{J}_{\alpha,\beta}(y) = \|y\|_{(m,\rho)}^2 \tilde{J}_{\alpha,\beta}\left(\frac{y}{\|y\|_{(m,\rho)}}\right) \geq \epsilon \|y\|_{(m,\rho)}^2$$

where $\epsilon = M(\alpha, \beta)$. Since $r_{\alpha,\beta}$ is Lipschitz continuous, as in Lemma 2.7, we have that $\|r_{\alpha,\beta}(y)\|_{(c,\sigma)} \leq C \|y\|_{(m,\rho)}$ for some $C > 0$, and we see that

$$\begin{aligned} I(u) &\geq \epsilon \|y\|_{(m,\rho)}^2 - \kappa \|u\|_{(m,\rho)} \\ &= \epsilon \|y\|_{(m,\rho)}^2 - \kappa \|r_{\alpha,\beta}(y) + y\|_{(m,\rho)} \\ &\geq \epsilon \|y\|_{(m,\rho)}^2 - \kappa (\|r_{\alpha,\beta}(y)\|_{(m,\rho)} + \|y\|_{(m,\rho)}) \\ &\geq \epsilon \|y\|_{(m,\rho)}^2 - \kappa (C + 1) \|y\|_{(m,\rho)} \end{aligned} \tag{4.3}$$

Thus, I is bounded below when restricted to \mathcal{Y} . □

As a consequence of the results above, there exists some $R > 0$ sufficiently large such that

$$\sup_{\{x \in X : \|x\|_{(c,\sigma)} = R\}} I(x) < \inf_{u \in \mathcal{Y}} I(u).$$

The next lemma shows the linking property of I . Let $B_R = \{x \in X : \|x\|_{(c,\sigma)} \leq R\}$ and $\partial B_R = \{x \in X : \|x\|_{(c,\sigma)} = R\}$.

Lemma 4.4. *Let $\gamma : B_R \subset X \rightarrow H^1(\Omega)$ be a continuous function such that $\gamma|_{\partial B_R}(x) = x$. Then $\gamma(B_R) \cap \mathcal{Y} \neq \emptyset$.*

Proof. Let $x \in B_R$ and let write $\gamma(x) = \gamma_X(x) + \gamma_Y(x)$, where $\gamma_X(x) \in X$ and $\gamma_Y(x) \in Y$. One can see that for all $x \in \partial B_R$, $\gamma_X(x) = x$ and $\gamma_Y(x) = 0$. To show that $\gamma(B_R) \cap \mathcal{Y} \neq \emptyset$, it suffices to show that there is an $x \in B_R$ such that $\gamma_X(x) = r_{\alpha,\beta}(\gamma_Y(x))$.

Let $H : B_R \rightarrow X$ defined by $H(x) = \gamma_X(x) - r_{\alpha,\beta}(\gamma_Y(x))$. We shall show that there is $x \in B_R$ such that $H(x) = 0$. Notice that H is continuous and for all $x \in \partial B_R$, $H(x) = x \neq 0$. Therefore, the Brouwer degree $deg(H, B_R, 0)$ is well defined. Now, consider the homotopy $h(x, t) = tH(x) + (1 - t)x$. Note that for $x \in \partial B_R$ we have $h(x, t) = tx + (1 - t)x = x \neq 0$. Hence, $deg(H, B_R, 0) = deg(Id, B_R, 0) = 1$, where Id represents the identity map. Thus $H(x) = 0$ has a solution in B_R . □

To prove Theorem 4.1 using the saddle point theorem of Rabinowitz, it suffices first to show I satisfies the Palais-Smale condition (PS) which builds some compactness into the functional I .

Lemma 4.5. *I satisfies the Palais-Smale condition (PS).*

Proof. Let $\{u_n\}$ be a sequence in $H^1(\Omega)$ such that $\{I(u_n)\}$ is bounded and $I'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. We will show that $\{u_n\}$ has a convergent subsequence. In view of the assumptions on the nonlinearities f and g , it suffices to first show that the sequence $\{u_n\}$ is bounded with respect to $\|\cdot\|_{(m,\rho)}$, that is, there exists a constant K such

that $\|u\|_{(m,\rho)} < K$. Suppose by contradiction that $\|u_n\|_{(m,\rho)} \rightarrow \infty$ as $n \rightarrow \infty$. Let $v_n = u_n/\|u_n\|_{(m,\rho)}$. Then

$$\frac{I(u_n)}{\|u_n\|_{(m,\rho)}^2} = J_{\alpha,\beta}(v_n) - \frac{1}{\|u_n\|_{(m,\rho)}^2} \left[\int F(x, u_n) + \oint G(x, u_n) \right].$$

Taking the limit, we have that $I(u_n)/\|u_n\|_{(m,\rho)}^2 \rightarrow 0$ since $\{I(u_n)\}$ is bounded, and $\frac{1}{\|u_n\|_{(m,\rho)}^2} [\int F(x, u_n) + \oint G(x, u_n)] \rightarrow 0$ because of the estimate (4.2). Hence, $I(u_n)/\|u_n\|_{(m,\rho)}^2$ and $[\int F(x, u_n) + \oint G(x, u_n)]/\|u_n\|_{(m,\rho)}^2$ are bounded. Also note that $\|v_n^\pm\|_{(m,\rho)} \leq 1$. From the definition of $J_{\alpha,\beta}$ it follows that $\|v_n\|_{(c,\sigma)}$ is bounded. Using the fact that $H^1(\Omega)$ is reflexive, the Sobolev compact embedding, and the continuity of the trace operator, we obtain that there exists a subsequence v_n that converges weakly to v_0 in $H^1(\Omega)$ and that converges strongly to v_0 in $L^2(\Omega)$ (also in $L^2(\partial\Omega)$). Since m and ρ are bounded functions and using the continuity of the norm $\|\cdot\|_{(m,\rho)}$, we obtain that $\|v_n\|_{(m,\rho)} \rightarrow \|v_0\|_{(m,\rho)}$. Thus, $\|v_0\|_{(m,\rho)} = 1$ since $\|v_n\|_{(m,\rho)} = 1$.

Now, for any $w \in H^1(\Omega)$,

$$\begin{aligned} \frac{I'(u_n)}{\|u_n\|_{(m,\rho)}} \cdot w &= \langle v_n, w \rangle_{(c,\sigma)} - \alpha \langle v_n^+, w \rangle_{(m,\rho)} + \beta \langle v_n^-, w \rangle_{(m,\rho)} \\ &\quad - \frac{1}{\|u_n\|_{(m,\rho)}} \left[\int m(x) \tilde{f}(u_n) w + \oint \rho(x) \tilde{g}(u_n) w \right] \end{aligned}$$

Using the boundedness of the nonlinearities f and g , and of the weights m and ρ , we have that

$$\frac{1}{\|u_n\|_{(m,\rho)}} \left[\int m(x) \tilde{f}(u_n) w + \oint \rho(x) \tilde{g}(u_n) w \right] \rightarrow 0.$$

Since $v_n \rightarrow v_0$ strongly in $L^2(\Omega)$ and $L^2(\partial\Omega)$, $\langle v_n^+, w \rangle_{(m,\rho)} \rightarrow \langle v_0^+, w \rangle_{(m,\rho)}$ and $\langle v_n^-, w \rangle_{(m,\rho)} \rightarrow \langle v_0^-, w \rangle_{(m,\rho)}$. By the weak convergence of v_n in $H^1(\Omega)$, we see that $\langle v_n, w \rangle_{(c,\sigma)} \rightarrow \langle v_0, w \rangle_{(c,\sigma)}$. We also note that $\frac{I'(u_n)}{\|u_n\|_{(m,\rho)}} \cdot w \rightarrow 0$ as $n \rightarrow \infty$. Hence,

$$0 = \langle v_0, w \rangle_{(c,\sigma)} - \alpha \langle v_0^+, w \rangle_{(m,\rho)} - \beta \langle v_0^-, w \rangle_{(m,\rho)} \quad \text{for all } w \in H^1(\Omega).$$

Thus v_0 is a nontrivial weak solution of (1.1). This leads to a contradiction since $(\alpha, \beta) \notin \Sigma$. Thus, $\{u_n\}$ are bounded with respect to $\|\cdot\|_{(m,\rho)}$.

Let us analyze carefully the functional I .

$$I(u_n) = \frac{1}{2} [\|u_n\|_{(c,\sigma)}^2 - \alpha \|u_n^+\|_{(m,\rho)}^2 - \beta \|u_n^-\|_{(m,\rho)}^2] - \left[\int F(x, u_n) + \oint G(x, u_n) \right]$$

Since $I(u_n)$ is bounded and using the fact that $\{u_n\}$ is bounded with respect to $\|\cdot\|_{(m,\rho)}$ and the estimate (4.2), we have that $\|u_n^+\|_{(m,\rho)}$, $\|u_n^-\|_{(m,\rho)}$, and $[\int F(x, u_n) + \oint G(x, u_n)]$ are all bounded. Thus, $\|u_n\|_{(c,\sigma)}$ must be bounded. Therefore there exists a subsequence u_n that converges weakly to u in $H^1(\Omega)$ and converges strongly to u in $L^2(\Omega)$ (also in $L^2(\partial\Omega)$). Since m and ρ are bounded functions and using the continuity of the norm $\|\cdot\|_{(m,\rho)}$, we obtain that $\|u_n\|_{(m,\rho)} \rightarrow \|u\|_{(m,\rho)}$.

Now, consider

$$\begin{aligned} I'(u_n) \cdot (u_n - u) &= \langle u_n, (u_n - u) \rangle_{(c,\sigma)} - \alpha \langle u_n^+, (u_n - u) \rangle_{(m,\rho)} + \beta \langle u_n^-, (u_n - u) \rangle_{(m,\rho)} \\ &\quad - \left[\int m(x) \tilde{f}(u_n) (u_n - u) + \oint \rho(x) \tilde{g}(u_n) (u_n - u) \right] \end{aligned}$$

By the assumption $I'(u_n) \rightarrow 0$ in $(H^1(\Omega))^*$ it follows that $I'(u_n).(u_n - u) \rightarrow 0$. Since $\|u_n^+\|_{(m,\rho)}$, $\|u_n^-\|_{(m,\rho)}$, \tilde{f} , and \tilde{g} are all bounded, and $\|u_n\|_{(m,\rho)} \rightarrow \|u\|_{(m,\rho)}$, we have that

$$\begin{aligned} & -\alpha \langle u_n^+, (u_n - u) \rangle_{(m,\rho)} + \beta \langle u_n^-, (u_n - u) \rangle_{(m,\rho)} \\ & - \left[\int m(x)f(u_n)(u_n - u) + \oint \rho(x)g(u_n)(u_n - u) \right] \rightarrow 0. \end{aligned}$$

Therefore $\langle u_n, (u_n - u) \rangle_{(c,\sigma)} \rightarrow 0$. Thus $\|u_n\|_{(c,\sigma)}^2 - \langle u_n, u \rangle_{(c,\sigma)} \rightarrow 0$.

Since u_n converges weakly to u in $H^1(\Omega)$, we have that $\langle u_n, u \rangle_{(c,\sigma)} \rightarrow \|u\|_{(c,\sigma)}^2$.

Hence, $\|u_n\|_{(c,\sigma)} \rightarrow \|u\|_{(c,\sigma)}$. Thus, $u_n \xrightarrow{(c,\sigma)} u$ in $H^1(\Omega)$. □

Proof of Theorem 4.1. The functional I satisfies the Palais-Smale condition due to Lemma 4.5, and by Lemma 4.4, I satisfies the linking property. Set

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in B_R \cap X} I(\gamma(u)),$$

where R is a sufficiently large constant, and $\Gamma = \{\gamma \in C(B_R \cap X; H^1(\Omega)) : \gamma|_{\partial B_R \cap X}(x) = x\}$. Then by the Saddle Point Theorem [9], it follows that c is a critical value of I . Thus problem (1.2) has a weak solution. □

5. RESONANCE PROBLEM

In this section, we again assume $(\alpha, \beta) \in \mathbb{R}^2$ with $\mu_k < \alpha < \mu_{k+1}$. However we now assume that $\beta = \beta(\alpha)$ so that $(\alpha, \beta) \in \Sigma$ by the characterization in Theorem 3.13. Again for notational convenience we take $X = X_k$. Most arguments from the previous section still apply, with the exception of Lemmas 4.3 part 2 and 4.5; namely that I is bounded from below and that I satisfies (PS). This is not surprising, as the case that $(\alpha, \beta) \in \Sigma$ corresponds to the case $\mu = \mu_{k+1}$ in the Fredholm alternative. We expect in such cases that solutions only exist when a generalized orthogonality condition is met.

In establishing existence of solutions in the non-resonance cases, we will need a generalized Landesman-Lazer condition, namely

Definition 5.1. If for any sequence $\{u_n\} \subset H^1(\Omega)$ such that $\|u_n\|_{(m,\rho)} \rightarrow \infty$ and $\frac{u_n}{\|u_n\|_{(m,\rho)}} \xrightarrow{(m,\rho)} \psi$, where ψ is a Fučík eigenfunction associated with (α, β) , we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} F(x, u_n) + \int_{\partial\Omega} G(x, u_n) = -\infty. \tag{5.1}$$

Theorem 5.2. Assume that $\mu_k < \alpha < \mu_{k+1}$, $\beta = \beta(\alpha)$, the nonlinearities f and g are bounded continuous functions, and $m \in L^\infty(\Omega)$ and $\rho \in L^\infty(\partial\Omega)$, then problem (1.2) has at least one weak solution provided that condition (5.1) holds.

Lemma 5.3. If (5.1) is satisfied, then I is bounded below on \mathcal{Y} .

Proof. Suppose to the contrary that there exists a sequence $\{u_n\} \subset \mathcal{Y}$ with $I(u_n) \rightarrow -\infty$. Since $\{u_n\} \subset \mathcal{Y}$, we may write $u_n = r_{\alpha,\beta}(y_n) + y_n$. Taking inequality (4.3) with $\epsilon = M(\alpha, \beta) = 0$, we observe that since $I(u_n) \rightarrow -\infty$, we must have $\|u_n\|_{(m,\rho)} \rightarrow \infty$. But since

$$\begin{aligned} \|u_n\|_{(m,\rho)}^2 &= \|r_{\alpha,\beta}(y_n) + y_n\|_{(m,\rho)}^2 \\ &= \|r_{\alpha,\beta}(y_n)\|_{(m,\rho)}^2 + \|y_n\|_{(m,\rho)}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\mu_1} \|r_{\alpha,\beta}(y_n)\|_{(c,\sigma)}^2 + \|y_n\|_{(m,\rho)}^2 \\
&\leq \frac{C^2}{\mu_1} \|y_n\|_{(m,\rho)}^2 + \|y_n\|_{(m,\rho)}^2 \\
&= \left(\frac{C^2}{\mu_1} + 1\right) \|y_n\|_{(m,\rho)}^2,
\end{aligned}$$

we observe that $\|y_n\|_{(m,\rho)} \rightarrow \infty$. Thus, no subsequence of $\{u_n\}$ lies in a set of the form $\{u \in \mathcal{Y} : u = r_{\alpha,\beta}(y) + y, \tilde{J}_{\alpha,\beta}(y) \geq c\|y\|\}$ for some $c > 0$, since if such a subsequence existed, this would imply $I(u) \rightarrow \infty$ by (4.3). Therefore, $\tilde{J}_{\alpha,\beta}(y_n) \rightarrow 0$ and by the homogeneity of $\tilde{J}_{\alpha,\beta}$, $\tilde{J}_{\alpha,\beta}\left(\frac{y_n}{\|y_n\|_{(m,\rho)}}\right) \rightarrow 0$. Since $M(\alpha, \beta) = 0$, $\{y_n/\|y_n\|_{(m,\rho)}\} \subset S_Y$ is a minimizing sequence of $\tilde{J}_{\alpha,\beta}$. As in the proof of Lemma 3.8, this implies that $\|y_n/\|y_n\|_{(m,\rho)}\|_{(c,\sigma)}$ is bounded. Therefore, there exists $y \in S_Y$ such that $y_n/\|y_n\|_{(m,\rho)} \xrightarrow{(c,\sigma)} y$ and $y_n/\|y_n\|_{(m,\rho)} \xrightarrow{(m,\rho)} y$. Using the homogeneity of $r_{\alpha,\beta}$, we have that

$$u_n = \|y_n\|_{(m,\rho)} \left(r_{\alpha,\beta}\left(\frac{y_n}{\|y_n\|_{(m,\rho)}}\right) + \frac{y_n}{\|y_n\|_{(m,\rho)}} \right),$$

and

$$\|u_n\|_{(m,\rho)} = \|y_n\|_{(m,\rho)} \left\| r_{\alpha,\beta}\left(\frac{y_n}{\|y_n\|_{(m,\rho)}}\right) + \frac{y_n}{\|y_n\|_{(m,\rho)}} \right\|_{(m,\rho)}.$$

Therefore,

$$\frac{u_n}{\|u_n\|_{(m,\rho)}} = \frac{r_{\alpha,\beta}\left(\frac{y_n}{\|y_n\|_{(m,\rho)}}\right) + \frac{y_n}{\|y_n\|_{(m,\rho)}}}{\left\| r_{\alpha,\beta}\left(\frac{y_n}{\|y_n\|_{(m,\rho)}}\right) + \frac{y_n}{\|y_n\|_{(m,\rho)}} \right\|_{(m,\rho)}}.$$

Therefore,

$$\begin{aligned}
\frac{u_n}{\|u_n\|_{(m,\rho)}} &\xrightarrow{(c,\sigma)} \frac{r_{\alpha,\beta}(y) + y}{\|r_{\alpha,\beta}(y) + y\|_{(m,\rho)}}, \\
\frac{u_n}{\|u_n\|_{(m,\rho)}} &\xrightarrow{(m,\rho)} \frac{r_{\alpha,\beta}(y) + y}{\|r_{\alpha,\beta}(y) + y\|_{(m,\rho)}}.
\end{aligned}$$

Setting

$$\frac{r_{\alpha,\beta}(y) + y}{\|r_{\alpha,\beta}(y) + y\|_{(m,\rho)}} = \phi.$$

we notice that $\|\phi\|_{(m,\rho)} = 1$ and $\tilde{J}_{\alpha,\beta}(\phi) = 0 = M(\alpha, \beta(\alpha))$. Therefore ϕ is a nontrivial eigenfunction associated to $(\alpha, \beta(\alpha))$. Since $\frac{u_n}{\|u_n\|_{(m,\rho)}} \xrightarrow{(m,\rho)} \phi$ and (5.1) is satisfied, we have that $\lim_{n \rightarrow \infty} (\int F(x, u_n) + \int G(x, u_n)) = -\infty$. It follows that $I(u_n) \rightarrow \infty$, a contradiction. The lemma is proved. \square

Lemma 5.4. *If (5.1) is satisfied, then I satisfies (PS).*

Proof. The first part of the proof is identical to the proof in Lemma 4.5. Suppose $\{u_n\} \subset H^1(\Omega)$ is a sequence such that $I(u_n)$ is bounded, $I'(u_n) \rightarrow 0$, and $\|u_n\|_{(m,\rho)} \rightarrow \infty$. As before, we take $v_n = \frac{u_n}{\|u_n\|_{(m,\rho)}}$ and by an identical argument we show that $v_n \xrightarrow{(c,\sigma)} v$ and $v_n \xrightarrow{(m,\rho)} v$ with $\|v\|_{(m,\rho)} = 1$ and v a Fučík eigenfunction associated with (α, β) . In the previous case this was a contradiction, but since $(\alpha, \beta) \in \Sigma$ in this case, we have not yet reached a contradiction, and further argument is needed.

Write $u_n = x_n + y_n = \tilde{x}_n + r_{\alpha,\beta}(y_n) + y_n$. Then

$$\begin{aligned} I'(u_n) \cdot \tilde{x}_n &= J'_{\alpha,\beta}(u_n) \cdot \tilde{x}_n - \int f(x, u_n) \tilde{x}_n + \oint g(x, u_n) \tilde{x}_n \\ &= J'_{\alpha,\beta}(\tilde{x}_n + r_{\alpha,\beta}(y_n) + y_n) \cdot \tilde{x}_n - \int f(x, u_n) \tilde{x}_n + \oint g(x, u_n) \tilde{x}_n \\ &= (J'_{\alpha,\beta}(\tilde{x}_n + r_{\alpha,\beta}(y_n) + y_n) - J'_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n)) \cdot \tilde{x}_n \\ &\quad - \int f(x, u_n) \tilde{x}_n + \oint g(x, u_n) \tilde{x}_n \\ &\leq -\delta \|\tilde{x}_n\|_{(c,\sigma)}^2 - \int f(x, u_n) \tilde{x}_n + \oint g(x, u_n) \tilde{x}_n \\ &\leq -\delta \mu_1 \|\tilde{x}_n\|_{(m,\rho)}^2 - \int f(x, u_n) \tilde{x}_n + \oint g(x, u_n) \tilde{x}_n \end{aligned}$$

by the fact that $J'_{\alpha,\beta}(r_{\alpha,\beta}(y) + y) \cdot x = 0$ for all $x \in X$ and Lemma 2.3. Dividing the inequality through by $\|\tilde{x}_n\|_{(m,\rho)}$ gives

$$I'(u_n) \cdot \frac{\tilde{x}_n}{\|\tilde{x}_n\|_{(m,\rho)}} \leq -\delta \mu_1 \|\tilde{x}_n\|_{(m,\rho)} - \int f(x, u_n) \tilde{x}_n + \oint g(x, u_n) \frac{\tilde{x}_n}{\|\tilde{x}_n\|_{(m,\rho)}},$$

but since $I'(u_n) \rightarrow 0$ and f, g are bounded, we obtain that $\|\tilde{x}_n\|_{(m,\rho)}$ is also bounded. It now follows that

$$J'_{\alpha,\beta}(u_n) \cdot \tilde{x}_n = I'(u_n) \cdot \tilde{x}_n + \int f(x, u_n) \tilde{x}_n + \oint g(x, u_n) \tilde{x}_n$$

must also be bounded.

Now, let $h(t) = J_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n + t\tilde{x}_n)$. Then $h'(t) = J'_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n + t\tilde{x}_n) \cdot \tilde{x}$, and we observe that $h'(0) = 0$ (by the definition of $r_{\alpha,\beta}$) and $h'(t)$ is decreasing by the strict concavity of $J_{\alpha,\beta}$ on $y_n + X$. By the Mean Value Theorem, $h(1) - h(0) = h'(c)$ for some $c \in (0, 1)$, and hence $h(1) - h(0) \geq h'(1)$ since h' is decreasing. So,

$$J_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n + \tilde{x}_n) - J_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n) \geq J'_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n + \tilde{x}) \cdot \tilde{x}_n.$$

Since $J_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n) \geq \|y_n\|_{m,\rho} M(\alpha, \beta) = 0$, we then have that $J_{\alpha,\beta}(u_n) \geq J'_{\alpha,\beta}(u_n) \cdot \tilde{x}_n$. Therefore,

$$\begin{aligned} I(u_n) &= J_{\alpha,\beta}(u_n) - \left[\int F(x, u_n) + \oint G(x, u_n) \right] \\ &\geq J'_{\alpha,\beta}(u_n) \cdot \tilde{x}_n - \left[\int F(x, u_n) + \oint G(x, u_n) \right]. \end{aligned}$$

However, $J'_{\alpha,\beta}(u_n) \cdot \tilde{x}_n$ is bounded and $[\int F(x, u_n) + \oint G(x, u_n)] \rightarrow -\infty$ by (5.1), which contradicts the boundedness of $I(u_n)$. Hence $\|u_n\|_{(m,\rho)}$ is bounded, and the proof proceeds as in Lemma 4.5. \square

By a straightforward application of the Saddle Point Theorem, we now conclude that there exists a solution to the resonance problem.

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