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On The Finiteness Of Quantum K-Theory Of A Homogeneous Space

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ON THE FINITENESS OF QUANTUM K-THEORY OF A HOMOGENEOUS SPACE

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ABSTRACT. We show that the product in the quantum K-ring of a generalized flag manifold G/P involves only finitely many powers of the Novikov variables. In contrast to previous approaches to this finiteness question, we exploit the finite difference module structure of quantum K-theory. At the core of the proof is a bound on the asymptotic growth of the J-function, which in turn comes from an analysis of the singularities of the zastava spaces studied in geometric representation theory.

An appendix by H. Iritani establishes the equivalence between finiteness and a quadratic growth condition on certain shift operators.

Let G be a simply connected complex semisimple group, with Borel subgroup B, maximal torus T, and standard parabolic group P. The main aim of this article is to prove a fundamental fact about the quantum K-ring of the homogeneous space G/P.

Theorem. *The structure constants for (small) quantum multiplication of Schubert classes in $QK_T(G/P)$ are polynomials in the Novikov variables, with coefficients in the representation ring of the torus.*

This is proved as Theorem 10 below. A priori, quantum structure constants are power series in the Novikov variables, which keep track of degrees of curves; our theorem says that in fact, only finitely many degrees appear. This property is often referred to as *finiteness* of the quantum product.

Finiteness has been the subject of conjectures since the beginnings of the combinatorial study of quantum K-theory in Schubert calculus. Indeed, this property is a foundational prerequisite for the main components of Schubert calculus: a presentation of the quantum K-ring as a quotient by a polynomial ring; a set of polynomial representatives for Schubert classes; and finally, combinatorial formulas for the structure constants themselves.

In quantum cohomology, finiteness of the quantum product is immediate from the definition. In this case, the structure constants are Gromov-Witten invariants—certain integrals on the moduli space of stable maps into G/P—and they automatically vanish for curves of sufficiently large degree, by dimension reasons. In K-theory, by contrast, the analogous Gromov-Witten invariants are certain Euler characteristics on the moduli space, and there is no reason for them to vanish for large degrees—in fact they do not.

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The structure constants for the quantum product in K-theory are rather complicated alternating sums of Gromov-Witten invariants, so a direct proof of finiteness involves demonstrating massive cancellation.

In the cases where finiteness was previously known, this direct approach was used, employing a detailed analysis of the geometry of the moduli space of stable maps, and especially its "Gromov-Witten subvarieties", whose Euler characteristics compute K-theoretic Gromov-Witten invariants of G/P. In their paper on Grassmannians, Buch and Mihalcea showed that these Gromov-Witten varieties are rational for sufficiently large degrees; this implies that the corresponding invariants are equal to 1, and the required cancellation can be deduced combinatorially [14]. Together with Chaput and Perrin, they extended this idea to prove finiteness for *cominuscule varieties*, a certain class of homogeneous varieties of Picard rank one [11, 12]. (Furthermore, according to [12, Remark 1.1], finiteness holds for any projective rational homogeneous space of Picard rank one.)

Recently, Kato [25, 26] has proven some remarkable conjectures [32] about the quantum K-ring of a *complete* flag variety G/B. Up to inverting some elements, he establishes ring isomorphisms

$$QK_T(G/B) \cong K_T^{\circ}(\text{ semi-infinite flag variety }) \cong K_{\circ}^{T}(\text{ affine Grassmannian }).$$

In particular, Kato's work implies finiteness for $QK_T(G/B)$. See [25, Corollary 3.3], noting that the argument given there relies on our Lemma 6 (in establishing the first isomorphism above), but otherwise is independent of our approach.

In this paper we prove the finiteness result for $QK_T(G/P)$ for all partial flag varieties. The starting point of our method is the fundamental fact that quantum K-theory admits the structure of a D_q-*module*. This structure was first found for the quantum K-theory of the complete flag variety $Fl_{r+1} = SL_{r+1}/B$ by Givental and Lee, and later derived in general by Givental and Tonita from a characterization theorem of quantum K-theory in terms of quantum cohomology, the so-called *quantum Hirzebruch-Riemann-Roch theorem* [20, 21]. As explained by Iritani, Milanov, and Tonita, this D_q-module structure is manifested as a difference equation (Equation (11) below) satisfied by certain generating functions J and T of K-theoretic Gromov-Witten invariants; they also explain how the quantum product by a line bundle is related to these generating functions and use this to compute the quantum product for Fl_3 [24]. More details are reviewed in §1.5.

The general strategy of our proof can be summarized as follows. If one can appropriately bound the coefficients appearing in the generating functions J and T, then results of [24] allow one to deduce that the quantum product by a line bundle is finite. For a complete flag variety, this is sufficient, since $K_T(G/B)$ is generated by line bundles. In fact, it is also true that the K-theory of G/P is generated by line bundle classes, after inverting certain elements of the representation ring; this seems to be less well known, so we include a proof in Lemma 1.

The technical heart of our argument lies in obtaining the appropriate bound on the growth of coefficients of J and T as $q \to +\infty$. Here we divide the problem and treat the G/B and G/P cases separately. For G/B, we analyze the geometry of the *zastava space*, a compactification of the space of (based) maps studied extensively in geometric representaion theory. Specifically, we use a computation of the canonical sheaf of the zastava space due to Braverman and Finkelberg [7, 8], together with some properties of its singularities. This leads to the bound for J stated in Lemma 4, as well as the stronger bound of Lemma 4^+ for simply-laced types. For bounds for T we appeal to Kato's work and a result of H. Iritani (the Proposition of Appendix B). We then transfer our bounds for G/B to bounds for G/P, using the main geometric constructions in Woodward's proof of the Peterson comparison formula [40].

With the bounds in hand, we deduce finiteness in §5. Here our arguments take advantage of the explicit form of our bounds for J, together with an inequality in root lattices proved in Appendix A.

We expect our methods to find further applications in quantum Schubert calculus. Most immediately, we can establish a presentation of the quantum K-ring of SL_{r+1}/B, resolving a conjecture by Kirillov and Maeno [36, 23]. (Using a different definition of quantum K-theory, a similar presentation was obtained in [29].) Together with algebraic work done by Ikeda, Iwao, and Maeno [23], this confirms some conjectural relations between the quantum K-ring of the flag manifold and the K-homology of the affine Grassmannian [32], giving an alternative to Kato's approach. Some results in this direction are included in our preprint [2].

A secondary goal of this article is to illustrate the power of finite-difference methods in quantum Schubert calculus. To this end, we have included a fair amount of background. We hope these sections may serve as a helpful companion to the foundational papers of Givental and others.

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1. Background

1.1. **Roots and weights.** Let Λ be the weight lattice for the torus T, and let $\varpi_1, \ldots, \varpi_r$ be the fundamental weights for the Lie algebra of G. The representation ring $R(T)$ is naturally identified with the group ring $\mathbb{Z}[\Lambda]$, and can be written as a Laurent polynomial ring $\mathbb{Z}[\mathrm{e}^{\pm\varpi_1}, \ldots, \mathrm{e}^{\pm\varpi_r}]$. The simple roots $\alpha_1, \ldots, \alpha_r$ generate a sublattice of Λ. The coroot lattice $\check{\Lambda}$ has a basis of simple coroots $\check{\alpha}_1, \ldots, \check{\alpha}_r$, dual to $\varpi_1, \ldots, \varpi_r$. We often write

$$\lambda = \lambda_1\varpi_1 + \cdots + \lambda_r\varpi_r \quad \text{and} \quad d = d_1\check{\alpha}_1 + \cdots + d_r\check{\alpha}_r$$

for elements of Λ and $\check{\Lambda}$. Usually, d denotes a *positive element* of the coroot lattice, meaning all the integers d_i are nonnegative. We write $d \geq 0$ or $d \in \check{\Lambda}_+$ to indicate positive elements, and $d > 0$ to mean a nonzero such d. This induces a partial order in the standard way, so $d' \geq d$ iff $d' - d \geq 0$; that is, $d'_i \geq d_i$ for all i.

The vector spaces $\Lambda \otimes \mathbb{R}$ and $\check{\Lambda} \otimes \mathbb{R}$ are identified using the Weyl-invariant inner product $(\ ,\ )$, normalized so that $(\alpha_i, \alpha_i) = 2$ when α_i is a short root. For example, this means $(d, \lambda) = \sum d_i \lambda_i$. For $G = SL_{r+1}$, we have

$$(d, d) = \sum_{i=1}^{r+1} (d_i - d_{i-1})^2,$$

where by convention $d_0 = d_{r+1} = 0$.

A *standard parabolic subgroup* is a closed subgroup P such that $G \supseteq P \supseteq B$. By recording which negative simple roots occurs as weights on the Lie algebra of P, such parabolics correspond to subsets of the simple roots. (To be clear, B corresponds to the empty set, while G corresponds to the whole set of simple roots.) Let $I_P \subseteq \{1, \ldots, r\}$ be the indices of simple roots corresponding to P.

The sublattice $\Lambda_P \subseteq \Lambda$ of weights λ such that $(\check{\alpha}_i, \lambda) = 0$ for $i \in I_P$ is spanned by the weights ϖ_j for $j \notin I_P$. Dually, $\check{\Lambda}_P \subseteq \check{\Lambda}$ is the sublattice spanned by $\check{\alpha}_i$ for $i \in I_P$. We write $\check{\Lambda}^P = \check{\Lambda}/\check{\Lambda}_P$, and $\check{\Lambda}^P_+$ for the image of $\check{\Lambda}_+$. So $\check{\Lambda}^P_+$ is spanned by the images of $\check{\alpha}_i$ for $i \notin I_P$.

Let $\rho = \varpi_1 + \cdots + \varpi_r$ be the *Weyl element*, the smallest regular dominant weight. For any $d \in \check{\Lambda}$, we have $(d, \rho) = \sum d_i =: |d|$.

1.2. **Flag varieties.** Each weight $\lambda \in \Lambda$ gives rise to an equivariant line bundle P^λ on the complete flag variety G/B. Writing P_i for the line bundle corresponding to ϖ_i, we have $P^\lambda = P_1^{\lambda_1} \cdots P_r^{\lambda_r}$ when $\lambda = \lambda_1 \varpi_1 + \cdots + \lambda_r \varpi_r$.

Each fundamental weight ϖ_i corresponds to an irreducible representation V_{ϖ_i}. There is an embedding

$$\iota \colon G/B \hookrightarrow \Pi := \prod_{i=1}^{r} \mathbb{P}(V_{\varpi_i}),$$

such that $P_i = \iota^* \mathcal{O}_i(-1)$ is the pullback of the tautological subbundle from the ith factor of Π.

For example, when $G = SL_{r+1}$, the flag variety $G/B = Fl_{r+1}$ parametrizes all complete flags in $\mathbb{C}^{r+1}$. We have $V_{\varpi_i} = \bigwedge^i \mathbb{C}^{r+1}$, and the line bundle P_i is the top exterior power $\bigwedge^i S_i$ of the ith tautological subbundle on Fl_{r+1}.[1]

Equivariant line bundles on G/P correspond to weights $\lambda \in \Lambda_P$. We will continue to use the notation P^λ for such bundles; the meaning of "P" (as parabolic or line bundle) should be clear from context. As with G/B, there is an embedding

$$\iota \colon G/P \hookrightarrow \prod_{j \notin I_P} \mathbb{P}(V_{\varpi_j}),$$

[1]Our conventions agree with [20], but are opposite to those of [24], where P_i is replaced by P_i^{-1}.

with P_j being the pullback of $\mathcal{O}(-1)$ from the jth factor.

There are natural identifications $H_2(G/B,\mathbb{Z}) = \check{\Lambda}$ and $\mathrm{Eff}_2(G/B) = \check{\Lambda}_+$, as well as $H_2(G/P,\mathbb{Z}) = \check{\Lambda}^P$ and $\mathrm{Eff}_2(G/P) = \check{\Lambda}^P_+$. The pushforward $H_2(G/B) \to H_2(G/P)$ is identified with the quotient map $\check{\Lambda} \to \check{\Lambda}^P$. The pullback $H^2(G/P) \to H^2(G/B)$ dual to this projection is identified with the inclusion $\Lambda_P \hookrightarrow \Lambda$.

It is a basic fact that $K_T(G/B)$ is generated by $P_1, \ldots, P_r$ as an $R(T)$-algebra; that is, there is a surjective homomorphism

$$R(T)[P_1, \ldots, P_r] \twoheadrightarrow K_T(G/B).$$

(See, for example, [30, §4].) Thus there is an $R(T)$-basis for $K_T(G/B)$ consisting of monomials in the P_i, and in particular, any other basis—for example, a Schubert basis—can be written as a finite $R(T)$-linear combination of such monomials.

In general, it is not the case that $K_T(G/P)$ is generated by line bundles as an $R(T)$-algebra. However, after extending scalars to the fraction field $F(T)$ of $R(T)$, the algebra is generated by line bundles. This fact seems to be less well known, although it is implicit in [13], and the idea of the proof can be found in [15, Lemma 4.1.3]. For clarity, we state a general version here.

Lemma 1. *Let $X \hookrightarrow Y$ be a closed T-equivariant inclusion of smooth varieties. Assume that the restriction homomorphism $K_T(Y^T) \to K_T(X^T)$ is surjective. If $\{\alpha\}$ is a set of generators for $K_T(Y)$ as an $R(T)$-algebra, then the restrictions $\{\beta\}$ generate $F(T) \otimes_{R(T)} K_T(X)$ as an $F(T)$-algebra.*

Proof. The proof follows directly from the localization theorem, which gives natural isomorphisms $F(T) \otimes_{R(T)} K_T(X) \cong F(T) \otimes_{R(T)} K_T(X^T)$. A little more precisely, rather than passing to $F(T)$, it suffices to invert elements $1 - \mathrm{e}^{-\alpha}$ of $R(T)$, where α runs over characters appearing in the normal spaces to X^T in X. □

A particular case of the lemma is this:

> *Whenever X is a smooth projective variety with finitely many attractive fixed points, the $F(T)$-algebra $F(T) \otimes_{R(T)} K_T(X)$ is generated by the class of an ample line bundle.*

An isolated fixed point p of a (possibly singular) variety X is called *attractive* if all the weights of the action of T on the Zariski tangent space at p lie in an open half-space. This condition guarantees that under any equivariant embedding $X \hookrightarrow \mathbb{P}^n$, each of the finitely many fixed points of X maps to a distinct connected component of $(\mathbb{P}^n)^T$, which in turn implies that the restriction map is surjective.

The standard torus action on G/P has finitely many attractive fixed points, so the lemma applies to the case we study. (A different, combinatorial argument for equivariant cohomology of G/P is given in [13, Remark 5.11].)

1.3. **Equivariant multiplicities and the fixed-point formula.** One of the main tools for computing in quantum K-theory is torus-equivariant localization on moduli spaces. We quickly review the main theorem we will use. This material is standard; see, e.g., [1] for an exposition aligned with our needs, [10] for a parallel discussion in the case of equivariant Chow groups, and [4] for applications to Gromov-Witten theory.

Suppose a torus T acts on a variety X. The Grothendieck group of equivariant coherent sheaves is $K_\circ^T(X)$. There is a natural isomorphism

$$F(T) \otimes_{R(T)} K_\circ^T(X^T) \xrightarrow{\sim} F(T) \otimes_{R(T)} K_\circ^T(X) \tag{1}$$

induced by pushforward from the fixed locus. (This goes back to Atiyah [3] and Quart [37].) Since T acts trivially on X^T, the left-hand side is

$$F(T) \otimes_{R(T)} K_\circ^T(X^T) = F(T) \otimes_{\mathbb{Z}} K_\circ(X^T) = \bigoplus_{Z \subseteq X^T} F(T) \otimes_{\mathbb{Z}} K_\circ(Z),$$

the direct sum over connected components $Z \subseteq X^T$.

If $Z \subseteq X^T$ is a connected component, the *equivariant multiplicity* of X along Z is the element $\varepsilon_Z(X)$ of $F(T) \otimes_{\mathbb{Z}} K_\circ(Z)$ defined so that

$$\sum_{Z \subseteq X^T} \varepsilon_Z(X) = [\mathcal{O}_X]$$

under the isomorphism (1). More generally, the localization isomorphism respects products by vector bundles: given a class $\xi \in K_T^\circ(X)$ (the Grothendieck group of equivariant vector bundles), one has

$$\sum_{Z \subseteq X^T} \varepsilon_Z(X) \cdot \xi|_Z = \xi \cdot [\mathcal{O}_X]. \tag{2}$$

Here $(\cdot)|_Z$ denotes the restriction homomorphism $K_T^\circ(X) \to K_T^\circ(Z)$.

The localization isomorphism is natural in an evident way: if $\pi \colon X \to Y$ is a proper equivariant morphism, then there is a commuting square

$$\begin{array}{ccc} F(T) \otimes_{R(T)} K_\circ^T(X^T) & \xrightarrow{\sim} & F(T) \otimes_{R(T)} K_\circ^T(X) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ F(T) \otimes_{R(T)} K_\circ^T(Y^T) & \xrightarrow{\sim} & F(T) \otimes_{R(T)} K_\circ^T(Y). \end{array}$$

Naturality immediately implies a useful formula for equivariant multiplicities. Assume $\pi_*[\mathcal{O}_X] = [\mathcal{O}_Y]$. (For example, this holds if X and Y both have rational singularities and π is birational, or has connected rational fibers.) Then for any connected component $W \subseteq Y^T$, we have the formula

$$\varepsilon_W(Y) = \sum_{Z \subseteq (\pi^{-1}W)^T} \pi_*^Z \varepsilon_Z(X), \tag{3}$$

the sum over connected components $Z \subseteq X^T$ which map into a given connected component $W \subseteq Y^T$, where $\pi^Z \colon Z \to W$ is the restriction of π. This gives a means of computing the equivariant multiplicities.

If the connected component $Z \subseteq X^T$ is regularly embedded, with conormal bundle $N^*_{Z/X}$, then the equivariant multiplicity is

$$\varepsilon_Z(X) = \frac{1}{\lambda_{-1}(N^*_{Z/X})}. \tag{4}$$

Here the denominator is the K-theoretic Euler class of $N_{Z/X}$. (More generally, for any vector bundle E of rank e, one defines

$$\lambda_{-1}(E) = 1 - E + \textstyle\bigwedge^2 E - \cdots + (-1)^e \textstyle\bigwedge^e E.)$$

Suppose $\pi \colon X \to Y$ is a proper equivariant map, and $W \subseteq Y^T$ is a connected component which is regularly embedded, such that all components $Z \subseteq (\pi^{-1}W)^T$ are regularly embedded in X. (For example, this happens automatically if X and Y are nonsingular varieties.) Combining (2), (3), and (4), we have

$$\frac{(\pi_*\xi)|_W}{\lambda_{-1}(N^*_{W/Y})} = \sum_{Z \subseteq (\pi^{-1}W)^T} \pi^Z_* \left(\frac{\xi|_Z}{\lambda_{-1}(N^*_{Z/X})} \right), \tag{5}$$

for any element $\xi \in K^T_\circ(X) = K_T^\circ(X)$,

A simple special case of the equivariant multiplicity will be of particular interest to us. When X is affine, and $Z = p$ is any fixed point, the equivariant multiplicity is equal to the *graded character* $\mathrm{ch}(\mathcal{O}_X)$ (see, e.g., [38]). If, furthermore, the fixed point is *attractive*, the equivariant multiplicity is equal to the multigraded Hilbert series of $\mathcal{O}_X$. (For example, if T acts on $X = \mathbb{A}^1$ by the character e^α, then it acts on $\mathcal{O}_X = \mathbb{C}[x]$ by scaling x by $\mathrm{e}^{-\alpha}$, so we have $\varepsilon_0(X) = \mathrm{ch}(\mathcal{O}_X) = 1/(1-\mathrm{e}^{-\alpha})$.)

1.4. **Quantum K-theory and moduli spaces.** The (genus 0) K-theoretic Gromov-Witten invariants are defined as certain sheaf Euler characteristics on the space of n-pointed, degree d stable maps,

$$\overline{M}_{0,n}(G/P, d).$$

This space comes with evaluation morphisms $\mathrm{ev}_i \colon \overline{M}_{0,n}(G/P,d) \to G/P$ for $1 \leq i \leq n$, which are equivariant for the action of T on G/P and the induced action on $\overline{M}_{0,n}(G/P,d)$. Given classes $\Phi_1, \ldots, \Phi_n \in K_T(G/P)$, there is a Gromov-Witten invariant

$$\chi(\overline{M}_{0,n}(G/P,d),\, \mathrm{ev}_1^*\Phi_1 \cdots \mathrm{ev}_n^*\Phi_n) \in R(T).$$

The *Novikov variables* keep track of curve classes in G/P; for $d \in \check{\Lambda}^P_+$, we write $Q^d = Q_1^{d_1} \cdots Q_r^{d_r}$. The (small) quantum K-ring of G/P is defined additively as

$$QK_T(G/P) := K_T(G/P) \otimes \mathbb{Z}[[Q]],$$

and is equipped with a *quantum product* $\star$ which deforms the usual (tensor) product on $K_T(G/P)$. Choosing any $R(T)$-basis[2] $\{\Phi_w\}$ for $K_T(G/P)$, and using the same notation for the corresponding $R(T)[[Q]]$-basis for $QK_T(G/P)$, one has

$$\Phi_u \star \Phi_v = \sum_{w,d} N_{u,v}^{w,d} Q^d \Phi_w,$$

where a priori the right-hand side is an infinite sum over all $d \in \check{\Lambda}_+^P$. (The structure constants $N_{u,v}^{w,d}$ are defined in a rather involved way via alternating sums of Gromov-Witten invariants; see [18, 34, 14] for details.)

We work mainly with two compactifications of the space $\mathrm{Hom}_d(\mathbb{P}^1, G/P)$ of degree d maps from $\mathbb{P}^1$ to G/P. The first is Drinfeld's *quasimap space* $\mathcal{Q}_d$, and we use it only for G/B. This space may be defined as follows; see, e.g., [5] for more details. For projective space $\mathbb{P}(V)$ and an integer $d_i \geq 0$, let $\mathbb{P}(V)_{d_i} = \mathbb{P}(\mathrm{Sym}^{d_i}\mathbb{C}^2 \otimes V)$ be the projective space of V-valued binary forms of degree d_i. (This is the quot scheme compactification of the space of degree d maps $\mathbb{P}^1 \to \mathbb{P}(V)$.) With $\Pi = \prod_{i=1}^r \mathbb{P}(V_{\varpi_i})$ as above and $d \in \check{\Lambda}_+$, let $\Pi_d = \prod_{i=1}^r \mathbb{P}(V_{\varpi_i})_{d_i}$. This contains the space of maps $\mathrm{Hom}_d(\mathbb{P}^1, \Pi)$ as an open subset. The embedding $\iota\colon G/B \hookrightarrow \Pi$ induces an embedding $\mathrm{Hom}_d(\mathbb{P}^1, G/B) \hookrightarrow \mathrm{Hom}_d(\mathbb{P}^1, \Pi)$, and the quasimap space $\mathcal{Q}_d$ is the closure of $\mathrm{Hom}_d(\mathbb{P}^1, G/B)$ inside Π_d.

Spaces of maps and quasimaps are equipped with a $\mathbb{C}^*$-action induced from an action on the source curve. The action on $\mathbb{P}^1$ is given by $q \cdot [a, b] = [a, qb]$, where q is a coordinate on $\mathbb{C}^*$, so the fixed points are $0 = [1, 0]$ and $\infty = [0, 1]$. The $\mathbb{C}^*$-fixed loci in Π_d are easy to describe: for each expression $d = d^- + d^+$ (with $d^-, d^+ \in \check{\Lambda}_+$), there is a fixed component $\Pi_d^{(d^+)}$ consisting of tuples of monomials of bidegree (d_i^-, d_i^+) on the factor $\mathbb{P}(V_{\varpi_i})_{d_i}$. Using monomials to denote weight bases for $\mathrm{Sym}^{d_i}\mathbb{C}^2$, we have

$$\Pi_d^{(d^+)} = \prod_{i=1}^r \mathbb{P}(x_i^{d_i^-} y_i^{d_i^+} \otimes V_{\varpi_i}),$$

so each such component is isomorphic to Π itself.

The $\mathbb{C}^*$-fixed components of $\mathcal{Q}_d \subseteq \Pi_d$ are $\mathcal{Q}_d^{(d^+)} \subseteq \Pi_d^{(d^+)}$, each isomorphic to $G/B \subseteq \Pi$.

If we also consider the action of T induced from the target space G/B, the quasimap space $\mathcal{Q}_d$ has finitely many $\mathbb{C}^* \times T$-fixed points, indexed by (d^+, w) as w ranges over the Weyl group.

Our second compactification of the space of maps is the *graph space*,

$$\Gamma(G/P)_d := \overline{M}_{0,0}(\mathbb{P}^1 \times G/P, (1, d)).$$

It includes $\mathrm{Hom}_d(\mathbb{P}^1, G/P)$ as the open subset of stable maps with irreducible source, regarded as the graph of a map $\mathbb{P}^1 \to G/P$. This space also comes with an action

[2]The classes Φ_w are not necessarily Schubert classes; in fact, after extending scalars from $R(T)$ to $F(T)$, we will use a monomial basis consisting of certain P^λ's.

of $\mathbb{C}^* \times T$, induced from the componentwise action on $\mathbb{P}^1 \times G/P$. As explained in [20, §2.2] and [24, §2.6], the $\mathbb{C}^*$-fixed components of $\Gamma(G/P)_d$ correspond to certain maps where the source curve is reducible. For each decomposition $d = d^- + d^+$, there is a component $\Gamma(G/P)_d^{(d^+)}$ whose general points parametrize maps with source curve having three components: a "horizontal" $\mathbb{P}^1$ with degree 0 with respect to G/P; a "vertical" $\mathbb{P}^1$ attached to the first component at the fixed point 0, with G/P-degree d^+; and a "vertical" $\mathbb{P}^1$ attached to the first component at ∞, with G/P-degree d^-. (If d^+ or d^- is zero, the corresponding component of the source curve is absent.) There are also pointed versions of graph spaces, $\Gamma(G/P)_{n,d}$, with $n \geq 0$ marked points, defined as $\overline{M}_{0,n}(\mathbb{P}^1 \times G/P, (1, d))$. The fixed loci of these pointed spaces are similar, with the marked points being allocated to one of the two vertical curves.

There is a birational morphism $\mu\colon \Gamma(G/B)_d \to \mathcal{Q}_d \subseteq \Pi_d$, described in [20, §3], and the fixed component $\Gamma(G/B)_d^{(d^+)}$ maps onto $\mathcal{Q}_d^{(d^+)}$ under μ. There are also morphisms $\beta_n\colon \Gamma(G/P)_{n,d} \to \overline{M}_{0,n}(G/P, d)$, which, composed with evaluation morphisms from $\overline{M}_{0,n}(G/P, d)$ to G/P, give morphisms $\mathrm{ev}_i\colon \Gamma(G/P)_{n,d} \to G/P$, for $1 \leq i \leq n$.

A key property of each of these moduli spaces—$\overline{M}_{0,n}(G/P, d)$, $\Gamma(G/P)_{n,d}$, and $\mathcal{Q}_d$—is that they have rational singularities. (For the first two, this is a general fact about varieties with finite quotient singularities; for $\mathcal{Q}_d$, it is one of the main theorems of [7, 8].) We exploit this to freely transport computations of Euler characteristics from one of these spaces to another.

1.5. **The J-function and D_q-module structure.** The structure of quantum K-theory becomes clearer when Gromov-Witten invariants are packaged into a generating function, the *J-function*. Note that the definitions of J vary somewhat in the literature. Ours is that of [20]; the function of [24] is equal to our $(1-q)J$. The function of [7] is a certain localization of our J-function. This function satisfies a finite-difference equation, and it is this D_q-module structure we exploit to prove finiteness of the quantum product. Here we review the properties of the J-function which we will need. In this subsection, X may be any smooth projective variety, as considered in [24].

Consider the evaluation morphism $\mathrm{ev}\colon \overline{M}_{0,1}(X, d) \to X$, which is equivariant for $\mathbb{C}^* \times T$ (with $\mathbb{C}^*$ acting trivially on both $\overline{M}_{0,1}(X, d)$ and X). The J-function is a power series in Q, with coefficients in $K_T(X) \otimes \mathbb{Q}(q)$:

$$J := 1 + \frac{1}{1-q} \sum_{d>0} Q^d \, \mathrm{ev}_* \left(\frac{1}{1-qL} \right). \tag{6}$$

Here the character q identifies $K_{\mathbb{C}^*}(\mathrm{pt}) = \mathbb{Z}[q^{\pm}]$, and L is the cotangent line bundle on $\overline{M}_{0,1}(X, d)$. (Its fiber at a moduli point $[f : (C, p) \to X]$ is T_p^*C.) We often write

$$J = \sum_{d \geq 0} J_d \, Q^d,$$

with $J_d \in K_T(X) \otimes \mathbb{Q}(q)$.

In [24], a *fundamental solution* T is defined. This is an element of $\mathrm{End}_{R(T)}(K_T(X))\otimes \mathbb{Q}(q)[[Q]]$, and is characterized by

(7)
$$\chi(X,\ \Phi_u\cdot\mathsf{T}(\Phi_v)) = \chi(X,\Phi_u\cdot\Phi_v)+\sum_{d>0} Q^d\chi\left(\overline{M}_{0,2}(X,d),\ \mathrm{ev}_1^*\Phi_u\cdot\frac{1}{1-qL_1}\cdot\mathrm{ev}_2^*\Phi_v\right),$$

for all Φ_u and Φ_v in an $R(T)$-basis of $K_T(X)$. Here L_1 is the cotangent line bundle at the first marked point of $\overline{M}_{0,2}(X,d)$. As with J, we write $\mathsf{T}=\sum_d Q^d\mathsf{T}_d$.

From the definitions of J and T, we see that J-function is recovered as $\mathsf{T}(1)$. (The factor of $1/(1-q)$ in the $d>0$ terms of J arises from the pushforward by the forgetful morphism $\overline{M}_{0,2}(X,d)\to\overline{M}_{0,1}(X,d)$, via the string equation in quantum K-theory; see [34, §4.4].)

Remark 2. Note that $\mathsf{T}|_{q=\infty} = \mathsf{T}|_{Q=0} = \mathrm{id}$. In particular, the expansion of T at $q=+\infty$ is of the form $\mathsf{T}=\mathrm{id}+O(q^{-1})$.

The coefficients J_d and the operators T_d can be computed by localization on the pointed graph space $\Gamma(X)_{n,d}$, and we mainly use this characterization. Consider the fixed component $\Gamma(X)_{n,d}^{(n,d)}$ which parametrizes stable maps in $\overline{M}_{0,n}(\mathbb{P}^1\times X,(1,d))$ whose source curve has a horizontal component of bi-degree $(1,0)$ and a vertical component of bi-degree $(0,d)$ attached to the horizontal component at 0, with all n marked points lying on the vertical component. The key is an identification

$$\Gamma(X)_{n,d}^{(n,d)}\cong\overline{M}_{0,n+1}(X,d)$$

obtained by taking account of the node at 0 where the vertical and horizontal components are attached.

Recall from §1.4 that $\mathbb{C}^*$ acts on $\Gamma(X)_{n,d}$ via its action on $\mathbb{P}^1$, by $q\cdot[a,b]=[a,qb]$, fixing $0=[1,0]$ and $\infty=[0,1]$. The normal bundle to the fixed component $\Gamma(X)_{n,d}^{(n,d)}$ has rank 2, and decomposes into a trivial line bundle of character q^{-1} (corresponding to moving the node away from 0 along the horizontal curve), and a copy of the tangent line bundle L_{n+1}^* on $\overline{M}_{0,n+1}(X,d)$ with character q^{-1} (corresponding to smoothing the node). (See, e.g., [20, p. 201], [7, Proof of Lemma 5.2], or [28, §1.3, §3.3].)

Now the localization formula (3) for the map $\mu_*\colon K_\circ^T(\Gamma(X)_d)\to K_\circ^T(\mathcal{Q}_d)$ says

$$\varepsilon_{\mathcal{Q}_d^{(d)}}(\mathcal{Q}_d)=\mu_*^{(d)}\left(\frac{1}{\lambda_{-1}(N^*)}\right) \tag{8}$$

where $\mu^{(d)}$ is the restriction of μ to the fixed component $\Gamma(X)_d^{(d)}$, N is the normal bundle to this component, and $\lambda_{-1}(N^*)=1-N^*+\bigwedge^2N^*-\cdots=(1-q)(1-qL)$. Using the identifications $\mathcal{Q}_d^{(d)}\cong X$, $\Gamma(X)_d^{(d)}\cong\overline{M}_{0,1}(X,d)$, and $\mu^{(d)}=\mathrm{ev}$, the right-hand side is exactly

$$J_d=\mathrm{ev}_*\left(\frac{1}{(1-q)(1-qL)}\right).$$

Similar reasoning identifies $\mathsf{T}_d(\xi)$ as

$$\frac{1}{1-q}T_d(\xi) = (\mathrm{ev}_1)_*\left(\frac{\mathrm{ev}_2^*\xi}{(1-q)(1-qL_1)}\right), \tag{9}$$

where we use the identification $\Gamma(X)_{1,d}^{(1,d)} \cong \overline{M}_{0,2}(X,d)$. See [20, §2.2 and §4.2].

Next we turn to the difference equations satisfied by J and T. The main theorems of [20], [7] say that J is an eigenfunction of the finite-difference Toda operator [16], [39], [17] when $X = G/B$ is a complete flag variety of type A, D, or E. (A modification of J satisfies the corresponding system in non-simply-laced types [8].) We only need part of this structure, which holds for general X, suitably interpreted as in [24]. To simplify the equations, we often write

$$\widetilde{J} = P^{\log Q/\log q}J \quad \text{and} \quad \widetilde{\mathsf{T}} = P^{\log Q/\log q}\mathsf{T},$$

where $P^{\log Q/\log q}$ means $P_1^{\log Q_1/\log q}\cdots P_r^{\log Q_r/\log q}$.

Consider the *q-shift operator* $q^{Q_i\partial_{Q_i}} : Q_j \mapsto q^{\delta_{ij}}Q_j$ which induces an action on power series in Q. The D_q-module structure of quantum K-theory has the following form.

For a finite sequence I consisting of integers $1 \le i \le r$,

$$\left(\prod_{i\in I} q^{Q_i\partial_{Q_i}}\right)\widetilde{J} = \widetilde{\mathsf{T}}\left(\prod_{i\in I}\mathsf{A}_r q^{Q_i\partial_{Q_i}}(1)\right). \tag{10}$$

where the A_i are certain operators in $\mathrm{End}_{R(T)}(K_T(X)) \otimes \mathbb{Q}[q][[Q]]$ defined in [24]; see especially [24, Proposition 2.10]. This is essentially a commutation relation between the operators $\widetilde{\mathsf{T}}$ and $q^{Q_i\partial_{Q_i}}$ which follows from [24, Remark 2.11]. Note that

$$\left(\prod_{i\in I} q^{Q_i\partial_{Q_i}}\right)P^{\log Q/\log q} = \left(\prod_{i\in I}P_i\right)P^{\log Q/\log q}\left(\prod_{i\in I} q^{Q_i\partial_{Q_i}}\right).$$

Cancelling a factor of $P^{\log Q/\log q}$ and noting that $\prod_{i\in I} q^{Q_i\partial_{Q_i}}$ operates by replacing J_d with $q^{\sum_{i\in I}d_i}J_d$, we can rewrite Equation (10) as

$$\prod_{i\in I}P_i\left(\sum_{d\ge 0} q^{\sum_{i\in I}d_i}J_dQ^d\right) = \mathsf{T}\left(\prod_{i\in I}\mathsf{A}_i q^{Q_i\partial_{Q_i}}(1)\right). \tag{11}$$

We can write $a_I := \prod_{i\in I}\mathsf{A}_i q^{Q_i\partial_{Q_i}}(1)$ as $a_I = \sum_{d\ge 0} a_I^{(d)}Q^d$ where each $a_I^{(d)}$ is polynomial in q by [24, Proposition 2.10]. As noted in Remark 2, $\mathsf{T} = \mathrm{id} + O(q^{-1})$, so we can rewrite Equation (11) as

$$\prod_{i\in I}P_i\left(1 + \sum_{d>0} q^{d_i}J_dQ^d\right) = \mathsf{T}(a_I) = a_I + \cdots = a_I^{(0)} + \sum_{d>0}a_I^{(d)}Q^d + \cdots, \tag{12}$$

where the omitted terms vanish at $q = \infty$.

Therefore the right-hand side of Equation (12)—namely, the leading terms of a_I—can be studied from the asymptotics of the left-hand side as $q \to +\infty$, specifically, the

$q^{\geq 0}$ coefficients of $q^{\sum_{i\in I} d_i} J_d$. We will see examples of how this works in Lemma 6 and Proposition 9 below.

2. THE ZASTAVA SPACE AND THE J-FUNCTION

To bound the degrees Q^d appearing in quantum products, our main tool will be a bound on the q-degree of the J-function and the operator T. To obtain the required bound, we need some technical properties of a slice of the quasimap space, called the *zastava space*. Definitions and detailed descriptions of this space can be found in [7], [9, §2], and [6]. (The last reference provides explicit coordinates.) We briefly review the main properties of the zastava space, and study a particular desingularization of it by the (Kontsevich) graph space.

2.1. **Singularities of the zastava space.** The zastava space $\mathcal{Z}_d$ is an affine variety which can be thought of as a compactification of based maps $(\mathbb{P}^1, \infty) \to (G/B, w_\circ)$. It is defined as a locally closed subvariety of $\mathcal{Q}_d$, as follows. Let $\mathcal{Q}_d^\circ$ be the open subset of quasimaps which have no "defect" at $\infty \in \mathbb{P}^1$; i.e., the locus parametrizing maps defined in a neighborhood of ∞. This comes with an evaluation morphism $\mathrm{ev}_\infty \colon \mathcal{Q}_d^\circ \to G/B$, and the zastava space is a fiber of this morphism: $\mathcal{Z}_d = \mathrm{ev}_\infty^{-1}(w_\circ)$. It has dimension $\dim \mathcal{Z}_d = 2|d| = (2\rho, d)$.

A key property of the zastava space is that it stratifies into smaller such spaces. Let $\mathcal{Z}_d^\circ = \mathcal{Z}_d \cap \mathrm{Hom}_d(\mathbb{P}^1, G/B)$ be the open set of based maps. Then

$$\mathcal{Z}_d = \coprod_{0\leq d'\leq d} \mathcal{Z}_{d'}^\circ \times \mathrm{Sym}^{d-d'}\mathbb{A}^1,$$

where for $e \in \check{\Lambda}_+$ the symmetric product $\mathrm{Sym}^e\mathbb{A}^1$ is a space of "colored divisors". Concretely, writing $e = e_1\check{\alpha}_1 + \cdots + e_r\check{\alpha}_r$ with each $e_i \in \mathbb{Z}_{\geq 0}$,

$$\mathrm{Sym}^e\mathbb{A}^1 = \prod_{i=1}^r \mathrm{Sym}^{e_i}\mathbb{A}^1.$$

For any $d' \leq d$, let $\partial_{d'}\mathcal{Z}_d \subseteq \mathcal{Z}_d$ be the closure of the stratum $\mathcal{Z}_{d-d'}^\circ \times \mathrm{Sym}^{d'}\mathbb{A}^1$. (See [7, §6]. By convention, let us declare $\partial_{d'}\mathcal{Z}_d$ to be empty if $d' \not\leq d$.) In particular, there are divisors $\partial_i\mathcal{Z}_d := \partial_{\check{\alpha}_i}\mathcal{Z}_d$.

We set

$$\Delta = \sum_{i=1}^r \partial_i \mathcal{Z}_d$$

and consider the pair $(\mathcal{Z}_d, \Delta)$. The strata of this pair can be described easily: for any $I \subseteq \{1, \ldots, r\}$, let

$$d_I = d - \sum_{i\in I} \check{\alpha}_i.$$

Then

$$\Delta_I := \bigcap_{i\in I} \partial_i \mathcal{Z}_d = \partial_{d_I}\mathcal{Z}_d,$$

understanding the RHS to be empty if $d_I \not\geq 0$.

Now consider the Kontsevich resolution of quasimaps by the graph space, $\Gamma(G/B)_d \to \mathcal{Q}_d$. This restricts to an equivariant resolution of the zastava space, which we write as $\phi\colon \widetilde{\mathcal{Z}}_d \to \mathcal{Z}_d$. Let $\widetilde{\Delta}$ be the proper transform of Δ under ϕ; this is a simple normal crossings divisor. Let $\widetilde{\omega}$ and ω be the canonical sheaves of $\widetilde{\mathcal{Z}}_d$ and $\mathcal{Z}_d$, respectively. Our goal is to show the following:

Proposition 3. *We have*

$$\phi_*\widetilde{\omega}(\widetilde{\Delta}) = \omega(\Delta), \quad \textit{and}$$
$$R^i\phi_*\widetilde{\omega}(\widetilde{\Delta}) = 0 \qquad \textit{for } i > 0.$$

In particular, $\phi_*[\widetilde{\omega}(\widetilde{\Delta})] = [\omega(\Delta)]$ *as classes in* $K_\circ^{\mathbb{C}^*\times T}(\mathcal{Z}_d)$.

Proof. We use the terminology and results of [27, §2.5]. In our context, this is the same as saying that $\phi\colon (\widetilde{\mathcal{Z}}_d, \widetilde{\Delta}) \to (\mathcal{Z}_d, \Delta)$ is a *rational resolution*. By [27, Proposition 2.84 and Theorem 2.87], it suffices to prove that the pair $(\mathcal{Z}_d, \Delta)$ is *dlt* and the resolution $\phi\colon (\widetilde{\mathcal{Z}}_d, \widetilde{\Delta}) \to (\mathcal{Z}_d, \Delta)$ is *thrifty*.

The fact that $(\mathcal{Z}_d, \Delta)$ is dlt is essentially proved in [7, 8]. In fact, the proof of [8, Proposition 5.2] shows that $(\mathcal{Z}_d, \Delta)$ is a klt pair, since $\omega(\Delta)$ is Cartier (in fact, trivial) and the relative log canonical divisor of the resolution ϕ has nonnegative coefficients. Since klt implies dlt, this suffices (see [27, Definition 2.8]).

The notion of a thrifty resolution $f\colon (Y, D_Y) \to (W, D)$ is defined in [27, Definition 2.79]: this means that W is normal, D is a reduced divisor, D_Y is the proper transform of D and has simple normal crossings, f is an isomorphism over the generic point of every stratum of the snc locus $\mathrm{snc}(W, D)$, and f is an isomorphism at the generic point of every stratum of (Y, D_Y).

The fact that $\phi\colon (\widetilde{\mathcal{Z}}_d, \widetilde{\Delta}) \to (\mathcal{Z}_d, \Delta)$ satisfies these conditions is straightforward. To check it, we review the description of ϕ, considering its values on strata. The component $\widetilde{\partial}_i$ is the proper transform of $\partial_i = \partial_i\mathcal{Z}_d \subseteq \mathcal{Z}_d$; a general point parametrizes stable maps whose source curve has a vertical component of degree $\check{\alpha}_i$, attached to a horizontal component of degree $d - \check{\alpha}_i$ at some point $x \neq \infty$. By remembering the map f from the horizontal component and the point x where the vertical component is attached, this maps to $(f, x) \in \mathcal{Z}^\circ_{d-\check{\alpha}_1} \times \mathbb{A}^1$.

Similarly, suppose $I = \{i_1, \ldots, i_k\}$ indexes a stratum. A general point of $\widetilde{\Delta}_I = \cap_{i\in I}\widetilde{\partial}_i$ consists of maps from a source curve with vertical components of degrees $\check{\alpha}_i$, one for each $i \in I$, attached to a horizontal component of degree $d' = d - \sum_{i\in I}\check{\alpha}_i$ at distinct points $x_{i_1}, \ldots, x_{i_k}$. This maps to $(f, x_{i_1}, \ldots, x_{i_k}) \in \mathcal{Z}^\circ_{d'} \times (\mathbb{A}^1)^k$, as before. Since the map $\widetilde{\mathcal{Z}}_{d'} \to \mathcal{Z}_d$ is birational, so is the map of strata $\widetilde{\Delta}_I \to \Delta_I$.

Finally, no subvariety of $\widetilde{\mathcal{Z}}_d$ other than $\widetilde{\Delta}_I$ maps onto the stratum Δ_I. Indeed, Δ_I is the closure of $\mathcal{Z}_{d'} \times (\mathbb{A}^1)^k$, with notation as in the previous paragraph, so a general point will have k distinct coordinates $x_{i_1}, \ldots, x_{i_k}$ for the $(\mathbb{A}^1)^k$ factor. The only preimage under ϕ of such a point is a map $(f, x_{i_1}, \ldots, x_{i_k})$ as described above.[3] □

2.2. **Asymptotics of the J-function.** A key ingredient in our approach to finiteness is a bound on the growth of the coefficients J_d, and more generally T_d, when considered as rational functions of q. Here we consider G/B; the extension to general G/P will be addressed later.

Given any $d \in \check{\Lambda}_+$, define

$$m_d := r(d) + \frac{(d,d)}{2}, \tag{13}$$

where $r(d)$ is the number of i such that $d_i > 0$.

Writing $J = \sum_d Q^d J_d$, each J_d is a rational function in q, with coefficients in $K_T(G/B)$. As $q \to \infty$, then, J_d tends to $c_d\, q^{-\nu_d}$, for some element $c_d \in K_T(G/B)$ and some integer ν_d.

Lemma 4. *We have $\nu_d \geq m_d$.*

Proof. Because $\mathbb{C}^*$ acts trivially on G/B, it is enough to compute the asymptotics of the restriction of J_d to any fixed point in $(G/B)^T$; we choose the point $w_\circ$, corresponding to the longest element of the Weyl group.

By Equation (8), the restriction $J_d|_{w_\circ}$ is equal to the contribution from the fixed point $(d, w_\circ) \in \mathcal{Q}_d^{\mathbb{C}^* \times T}$ appearing in the localization formula for $\chi(\mathcal{Q}_d,\, \mathcal{O})$. The localization formula (3), applied to the map $\mathcal{Q}_d \to \mathrm{pt}$, says

$$\chi(\mathcal{Q}_d,\, \mathcal{O}) = \sum_{(d^+,w)} \varepsilon_{(d^+,w)}(\mathcal{Q}_d).$$

So we only need to compute the equivariant multiplicity, or more specifically, its degree as a rational function in q.

We may reduce to the zastava space $\mathcal{Z}_d$; from its description as the fiber over $w_\circ \in G/B$ of the evaluation map $\mathrm{ev}_\infty \colon \mathcal{Q}_d^\circ \to G/B$, we see that

$$\varepsilon_{(d,w_\circ)}(\mathcal{Q}_d) = \left(\prod \frac{1}{1 - \mathrm{e}^{-\alpha}} \right) \cdot \varepsilon_0(\mathcal{Z}_d),$$

where the product is over positive roots α. In particular, the contribution of q to $\varepsilon_{(d,w_\circ)}(\mathcal{Q}_d)$ comes from $\varepsilon_0(\mathcal{Z}_d)$, so it is enough to compute the latter.

[3]There are other subvarieties of $\widetilde{\mathcal{Z}}_d$ mapping into Δ_I, but not dominantly. For instance, there is a divisor $D_{\check{\alpha}_1+\check{\alpha}_2} \subseteq \widetilde{\mathcal{Z}}_d$ where the source curve has a vertical component of degree $\check{\alpha}_1 + \check{\alpha}_2$ attached at a point x to a horizontal component of degree $d - \check{\alpha}_1 - \check{\alpha}_2$. This maps to $\partial_1 \cap \partial_2$, but in the stratum $\mathcal{Z}^\circ_{d-\check{\alpha}_1-\check{\alpha}_2} \times (\mathbb{A}^1)^2$, the image only contains points in the diagonal $\mathbb{A}^1 = \{(x,x)\} \subseteq (\mathbb{A}^1)^2$.

Let us write

$$\varepsilon_0(\mathcal{Z}_d) = \frac{R(q)}{S(q)}$$

as a rational function in q. We wish to show

$$\deg(R) - \deg(S) \leq -m_d = -r(d) - \frac{(d,d)}{2}, \tag{14}$$

or in other words, the order of the rational function is $\mathrm{ord}_\infty(\varepsilon_0(\mathcal{Z}_d)) \geq m_d$. This will give the asserted bound.

Using the notation of Proposition 3, recall $\omega = \omega_{\mathcal{Z}_d}$ is the canonical sheaf, and $\Delta \subseteq \mathcal{Z}_d$ is the boundary divisor. By the proof of [8, Proposition 5.2], $\omega(\Delta)$ is a trivial line bundle, with q-weight $(d,d)/2 = m_d - r(d)$, so

$$\mathrm{ch}(\omega(\Delta)) = q^{m_d - r(d)}\,\varepsilon_0(\mathcal{Z}_d). \tag{15}$$

We will show that the rational function $\mathrm{ch}(\omega(\Delta))$ has $\mathrm{ord}_\infty(\mathrm{ch}(\omega(\Delta))) \geq r(d)$, which proves Equation (14) after dividing by $q^{m_d - r(d)}$.

To see this, we will compute $\mathrm{ch}(\omega(\Delta))$ by localization, using the Kontsevich resolution and the identity $[\omega(\Delta)] = \phi_*[\widetilde{\omega}(\widetilde{\Delta})]$ from Proposition 3. Recalling the descriptions of the $\mathbb{C}^*$-fixed components of $\Gamma(G/B)_d$, one sees that $\widetilde{\mathcal{Z}}_d$ has a unique fixed component, namely

$$\mathcal{F} = \widetilde{\mathcal{Z}}_d^{\mathbb{C}^*} = \Gamma(G/B)_d^{(d)} \cap \widetilde{\mathcal{Z}}_d.$$

A general point parametrizes based maps where the source curve consists of a horizontal component of degree 0 (mapping to $w_\circ \in G/B$) with a vertical component of degree d, attached to the horizontal component at the fixed point 0.

Now we have

$$\mathrm{ch}(\omega(\Delta)) = \varepsilon_0(\mathcal{Z}_d) \cdot [\omega(\Delta)]|_0 = \phi_*\left(\frac{\widetilde{\omega}(\widetilde{\Delta})|_{\mathcal{F}}}{\lambda_{-1}(N^*_{\mathcal{F}/\widetilde{\mathcal{Z}}_d})}\right). \tag{16}$$

Taking q-graded characters, the fraction in the right-hand side has order $r(d)$ at $q = \infty$. Indeed, the nontrivial characters appearing in $\widetilde{\omega}|_{\mathcal{F}}$ are precisely those appearing as normal characters in $N_{\mathcal{F}/\widetilde{\mathcal{Z}}_d}$. (The tangential directions along $\mathcal{F}$ have trivial character, since $\mathcal{F}$ is fixed.) Each irreducible component of the divisor $\widetilde{\Delta}$ contributes q^{-1}, by the proof of [7, Lemma 5.2], and there are $r(d)$ such components. Finally, after pushing forward by ϕ, we see that the order at ∞ of the right-hand side is at least $r(d)$. (Some terms may vanish in the pushforward, so inequality is possible.) □

In the case where G is simply laced—i.e., of type A, D, or E—a similar (but simpler) argument produces a stronger bound. Let $k_d := (\rho, d) + \frac{(d,d)}{2}$.

Lemma 4$^+$. *When G is simply laced, we have $\nu_d \geq k_d$.*

Proof. The argument is exactly as before, with the following changes. First, we have that ω itself is a trivial line bundle with character $q^{(\rho,d)+(d,d)/2}$, as in the proof of [7, Lemma 5.2], so that

$$\mathrm{ch}(\omega) = q^{k_d}\,\varepsilon_0(\mathcal{Z}_d).$$

Next, we have $\phi_*[\widetilde{\omega}] = [\omega]$ using the fact that $\mathcal{Z}_d$ has rational singularities [7, Proposition 5.1]. Finally, the fraction

$$\frac{\widetilde{\omega}|_{\mathcal{F}}}{\lambda_{-1}(N^*_{\mathcal{F}/\widetilde{\mathcal{Z}}_d})}$$

has order 0 at infinity, so pushing forward by ϕ shows that $\mathrm{ord}_\infty(\mathrm{ch}(\omega)) \geq 0$. Dividing by q^{k_d} yields the bound. □

Remark. In type A, the exponent is

$$k_d = d_1 + \cdots + d_r + \sum_{i=1}^{r+1} \frac{(d_i - d_{i-1})^2}{2},$$

where $d_0 = d_{r+1} = 0$, which agrees with [20, Eq. (7)].

Remark. For any smooth projective variety X, using the characterization of J_d as

$$J_d = \mathrm{ev}_* \left(\frac{1}{(1-q)(1-qL)} \right),$$

where $\mathrm{ev}\colon \overline{M}_{0,1}(X,d) \to X$ is the evaluation, one can interpret ν_d as the minimal integer ≥ 2 such that $\mathrm{ev}_*(L^{-\nu_d+1}) \neq 0$ in $K_T(X)$. Indeed, one expands this pushforward in powers of q^{-1} as

$$q^{-2}(1 + q^{-1} + q^{-2} + \cdots)\mathrm{ev}_* \left(L^{-1}(1 + q^{-1}L^{-1} + q^{-2}L^{-2} + \cdots) \right).$$

A similar characterization of the order of T_d at $q = \infty$ will be useful below.

2.3. **Comparison between the Borel and parabolic cases.** We compare the vanishing orders (at $q = \infty$) of T for G/B and G/P. Our main tool is a construction due to Woodward, in the course of his proof of the Peterson-Woodward comparison formula relating quantum cohomology of G/P to that of G/B [40].

Given any $d_P \geq 0$ in $\check{\Lambda}^P$, the Peterson-Woodward formula produces another parabolic P', with $P \supseteq P' \supseteq B$, together with canonical lifts $d_{P'} \in \check{\Lambda}^{P'}_+$ and $d_B \in \check{\Lambda}_+$ of d_P. There are natural morphisms

$$h_{P'/B}\colon \Gamma(G/B)_{n,d_B} \to \Gamma(G/P')_{n,d_{P'}} \times_{G/P'} G/B$$

and

$$h_{P/P'}\colon \Gamma(G/P')_{n,d_{P'}} \to \Gamma(G/P)_{n,d_P},$$

where $\Gamma(G/B)_{n,d_B} \to \Gamma(G/P')_{n,d_{P'}}$ and $\Gamma(G/P')_{n,d_{P'}} \to \Gamma(G/P)_{n,d_P}$ come from functoriality of the Kontsevich space, and $\Gamma(G/B)_{n,d_B} \to G/B$ and $\Gamma(G/P')_{n,d_{P'}} \to G/P'$ are given by evaluation at $0 \in \mathbb{P}^1$. (This makes sense, since any source curve in the graph space has a distinguished component together with a fixed isomorphism to $\mathbb{P}^1$.)

Woodward shows that these morphisms are birational. More precisely, [40, Theorem 3] asserts that the corresponding maps between Hom spaces are birational, and these are dense open sets in our graph spaces.

Explicit formulas for d_B and P' can be found in [33, Remark 10.17], but for our purposes it is enough to know that d_B and $d_{P'}$ map to d_P under the canonical projection, and that the above birational morphisms exist.

Consider $d_P \geq 0$ in $\check{\Lambda}^P$, and let us define ν_{d_P} as for the G/B case: it is the exponent so that J_{d_P} tends to $c_{d_P}\, q^{-\nu_{d_P}}$ as $q \to \infty$, for some $c_{d_P} \in K_T(G/P)$. In other words, $\nu_{d_P} = \mathrm{ord}_\infty(J_{d_P})$.

In addition to the Peterson-Woodward lift d_B of a degree $d_P \in \check{\Lambda}^P_+$, there is another canonical lift, which we call the *minimal* lift $d_B^{\min} \in \check{\Lambda}_+$. This is (unique) smallest effective lift of d_P. Explicitly, write $d_P = \sum c_i \bar{\alpha}_i$, where the sum is over $i \notin I_P$, each $c_i \geq 0$, and $\bar{\alpha}_i$ is the image of $\check{\alpha}_i$ in $\check{\Lambda}^P$. Then $d_B^{\min} = \sum c_i \check{\alpha}_i$.

Here is the main lemma of this section.

Lemma 5. *For any $\xi \in K_T(G/P)$, we have*

$$\mathrm{ord}_{q=\infty} \mathsf{T}_{d_P}(\xi) \geq \min_{d_B^{\min} \leq d_B^+ \leq d_B} \{\mathrm{ord}_{q=\infty} \mathsf{T}_{d_B^+}(\pi^* \xi)\},$$

where $\pi\colon G/B \to G/P$ is the projection. In particular, taking $\xi = 1$, we have $\nu_{d_P} \geq \min_{d_B^{\min} \leq d_B^+ \leq d_B} \{m_{d_B^+}\}$.

Proof. When $\xi = 1$, the displayed inequality is precisely $\nu_{d_P} \geq \min_{d_B^{\min} \leq d_B^+ \leq d_B} \{\nu_{d_B^+}\}$, so the second statement follows from Lemma 4.

To verify $\mathrm{ord}_{q=\infty} \mathsf{T}_{d_P}(\xi) \geq \mathrm{ord}_{q=\infty} \mathsf{T}_{d_B^{\min}}(\pi^*\xi)$, we use the characterization $\mathsf{T}_d(\xi) = (\mathrm{ev}_1)_* \left(\frac{\mathrm{ev}_2^* \xi}{1 - qL_1}\right)$ from Equation (9), where $\mathrm{ev}_i\colon \overline{M}_{0,2}(X,d) \to X$ are the evaluation maps. Let

$$h\colon \Gamma(G/B)_{n,d_B} \to \Gamma(G/P)_{n,d_P}$$

be the composition of $h_{P'/B}$, the projection on the first factor, and $h_{P/P'}$. The $\mathbb{C}^*$-fixed loci of $\Gamma(G/B)_{n,d_B}$ which map to the fixed component $\Gamma(G/P)^{(1,d_P)}_{1,d_P}$ are the components $\Gamma(G/B)^{(1,d_B^+)}_{1,d_B}$ such that $d_B^{\min} \leq d_B^+ \leq d_B$. Recall the identifications of fixed loci

$$\Gamma(G/B)^{(1,d_B^+)}_{1,d_B} \cong \overline{M}_{0,2}(G/B, d_B^+) \times_{G/B} \overline{M}_{0,1}(G/B, d_B^-) \quad \text{and}$$
$$\Gamma(G/P)^{(1,d_P)}_{1,d_P} \cong \overline{M}_{0,2}(G/P, d_P),$$

where in the fiber product both maps to G/B are by ev_1. We have commutative diagrams

$$\begin{array}{ccccccc}
G/B & \xleftarrow{\mathrm{ev}_1} & \overline{M}_{0,2}(G/B, d_B^+) \times_{G/B} \overline{M}_{0,1}(G/B, d_B^-) & \xhookrightarrow{\iota} & \Gamma(G/B)_{1,d_B} & \xrightarrow{\mathrm{ev}} & G/B \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \bar{h}^{d_B^+}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \pi} \\
G/P & \xleftarrow{\mathrm{ev}_1} & \overline{M}_{0,2}(G/P, d_P) & \xhookrightarrow{\iota} & \Gamma(G/P)_{1,d_P} & \xrightarrow{\mathrm{ev}} & G/P
\end{array}$$

for each such d_B^+, where $d_B = d_B^+ + d_B^-$. In the bottom row, the composition $\mathrm{ev} \circ \iota$ is equal to $\mathrm{ev}_2 \colon \overline{M}_{0,2}(G/P, d_P) \to G/P$, and similarly in the top row (when one also composes with the projection on the first factor). Since h is the composition of birational morphisms between varieties with rational singularities and a smooth projection with rational fibers, we have $h_*h^*(z) = z$ for any $z \in K_T(\Gamma(G/P)_{1,d_P})$. Furthermore, by the localization formula (3) applied to $\bar{h}$, for any $\alpha \in K_T(\Gamma(G/B)_{1,d_B})$ we have

$$\begin{aligned}\frac{\iota^* h_*(\alpha)}{(1-q)(1-qL_1^P)} &= \bar{h}_*^{d_B}\left(\frac{\iota^*\alpha}{(1-q)(1-qL_1)}\right) \\ &\quad + \sum_{d_B^{\min} \le d_B^+ < d_B} \bar{h}_*^{d_B^+}\left(\frac{\iota^*\alpha}{(1-q)(1-qL_1)(1-q^{-1})(1-q^{-1}L')}\right).\end{aligned}$$

Here L_1^P is the cotangent line bundle at the first marked point of $\overline{M}_{0,2}(G/P, d_P)$, and L_1 and L' are the pullbacks of cotangent line bundles on $\overline{M}_{0,2}(G/B, d_B^+)$ and $\overline{M}_{0,1}(G/B, d_B^-)$, respectively. (The denominators are the K-theoretic top Chern classes of the normal bundles to the respective fixed loci, see e.g. [24, §2.6].)

Now we set $\alpha = \mathrm{ev}^*\pi^*\xi = h^*\mathrm{ev}^*\xi$ in the above equation and apply $(\mathrm{ev}_1)_*$ to both sides. On the left-hand side, we obtain

$$(\mathrm{ev}_1)_*\left(\frac{\iota^* h_* h^* \mathrm{ev}^*\xi}{(1-q)(1-qL_1^P)}\right) = (\mathrm{ev}_1)_*\left(\frac{\mathrm{ev}_2^*\xi}{(1-q)(1-qL_1^P)}\right) = \frac{1}{1-q} T_{d_P}(\xi).$$

For the first term on the right-hand side, we compute

$$\begin{aligned}(\mathrm{ev}_1)_*\bar{h}_*^{d_B}\left(\frac{\iota^* h^* \mathrm{ev}^*\xi}{(1-q)(1-qL_1)}\right) &= \pi_*(\mathrm{ev}_1)_*\left(\frac{\iota^*\mathrm{ev}^*\pi^*\xi}{(1-q)(1-qL_1)}\right) \\ &= \pi_*(\mathrm{ev}_1)_*\left(\frac{\mathrm{ev}_2^*\pi^*\xi}{(1-q)(1-qL_1)}\right) \\ &= \frac{1}{1-q}\pi_*\mathsf{T}_{d_B}(\pi^*\xi),\end{aligned}$$

so this term vanishes to order at least $\mathrm{ord}_{q=\infty}\,\mathsf{T}_{d_B}(\pi^*\xi)$.

The other terms are similar. Writing $\mathrm{pr}\colon \overline{M}_{0,2}(G/B, d_B^+) \times_{G/B} \overline{M}_{0,1}(G/B, d_B^-) \to \overline{M}_{0,1}(G/B, d_B^-)$ for the second projection, we have

$$(\mathrm{ev}_1)_* \bar{h}_*^{d_B^+} \left(\frac{\iota^* h^* \mathrm{ev}^* \xi}{(1-q)(1-qL_1)(1-q^{-1})(1-q^{-1}L')} \right)$$
$$= \frac{1}{(1-q)(1-q^{-1})} \pi_* (\mathrm{ev}_1)_* \left(\mathrm{pr}_* \left(\frac{\iota^* \mathrm{ev}^* \pi^* \xi}{1-qL_1} \right) \cdot \frac{1}{1-q^{-1}L'} \right)$$
$$= \frac{q^{-2}}{(1-q^{-1})^2} \pi_* (\mathrm{ev}_1)_* \left(\mathrm{pr}_* \left(\mathrm{ev}_2^* \pi^* \xi \cdot L_1^{-1}(1 + q^{-1}L_1^{-1} + q^{-2}L_1^{-2} + \cdots) \right) \cdot \frac{1}{1-q^{-1}L'} \right).$$

The factor

$$\frac{q^{-2}}{(1-q^{-1})^2} \mathrm{pr}_* \left(\mathrm{ev}_2^* \pi^* \xi \cdot L_1^{-1}(1 + q^{-1}L_1^{-1} + q^{-2}L_1^{-2} + \cdots) \right)$$

has vanishing order equal to $\mathrm{ord}_{q=\infty} \mathsf{T}_{d_B^+}(\pi^*\xi)$, since pr is a flat pullback of the evaluation map $\mathrm{ev}_1\colon \overline{M}_{0,2}(G/B, d_B^+) \to G/B$ which computes $\mathsf{T}_{d_B^+}$. So the whole term vanishes at least to order $\mathrm{ord}_{q=\infty} \mathsf{T}_{d_B^+}(\pi^*\xi)$. Our claim follows. □

When G is simply laced, the same argument produces a sharper bound:

Lemma 5$^+$. *If G is simply laced, we have $\nu_{d_P} \geq \min_{d_B^{\min} \leq d_B^+ \leq d_B} \{k_{d_B^+}\}$.* □

Remark. For degrees d_P such that $d_B = d_B^{\min}$, the same argument shows that

$$\mathsf{T}_{d_P}(\xi) = \pi_* \mathsf{T}_{d_B}(\pi^*\xi),$$

since in this case there is only one term in the localization formula.

3. The operator $\mathsf{A}_{i,\mathrm{com}}$

For a partial flag variety G/P and a degree $d = d_P$, we write $\hat{d}$ for an associated degree on G/B which lies in the interval between $d_B^{\min}$ and d_B, and achieves the minimum of $m_{d_B^+}$ among degrees d_B^+ in this interval, as in as in §2.3. That is,

$$m_{\hat{d}} = \min_{d_B^{\min} \leq d_B^+ \leq d_B} \{m_{d_B^+}\},$$

and by Lemma 5, we have $\nu_d \geq m_{\hat{d}}$.

As discussed in §1.5, certain operators $\mathsf{A}_i \in \mathrm{End}_{R(T)}(K_T(G/P)) \otimes \mathbb{Q}[q][[Q]]$, defined and studied in [24], give the D_q-module structure of quantum K-theory.

Setting $q = 1$ in A_i produces operators $\mathsf{A}_{i,\mathrm{com}} := \mathsf{A}_i|_{q=1} \in \mathrm{End}(K_T(G/P)) \otimes \mathbb{Q}[[Q]]$. By Equation (11) and [24, Proposition 2.12], we have

$$\prod_{i \in I} \mathsf{A}_{i,\mathrm{com}}(1) = a_I|_{q=1} \tag{17}$$

Lemma 6. *The operator $\mathsf{A}_{i,\mathrm{com}}$ is the operator of the (small) quantum product by P_i.*

Proof. It suffices to show that $\mathsf{A}_{i,\mathrm{com}}(1) = P_i$. By [24, Proposition 2.10], the operators $\mathsf{A}_{i,\mathrm{com}}$ act as the (small) quantum product: we have

$$\mathsf{A}_{i,\mathrm{com}}(\Phi) = \left(P_i + \sum_{d>0} c_{d,i} Q^d\right) \star \Phi, \tag{18}$$

for some $c_{d,i} \in K_T(G/P)$. We will prove that $c_{d,i} = 0$ for all $d > 0$.

Writing $a = \mathsf{A}_i q^{Q_i \partial_{Q_i}}(1)$ and applying Equation (12), we obtain

$$P_i \left(1 + \sum_{d>0} q^{d_i} J_d Q^d\right) = \mathsf{T}(a) = a^{(0)} + \sum_{d>0} a^{(d)} Q^d + \cdots , \tag{19}$$

where the omitted terms vanish at $q = \infty$.

As in the discussion after Equation (12), we compute $\mathsf{A}_{i,\mathrm{com}}(1) = a|_{q=1}$ by studying the asymptotics of the expansion of the left-hand side of (19) at $q = \infty$.

To prove the lemma, we wish to show $a^{(d)} = 0$ for $d > 0$. For this, since we know that $a^{(d)}$ is polynomial in q, it suffices to show that there are no $q^{\geq 0}$ coefficients of Q^d on the left-hand side of (19).

Suppose a $d > 0$ term contributes to the $q^{\geq 0}$ coefficients—that is, suppose $q^{d_i} J_d$ has non-positive order at $q = \infty$. This means that $d_i \geq \nu_d$. Noting that $\hat{d}_i = d_i$ since $\hat{d} = d_B$ is a lift of $d = d_P$, Lemma 5 gives

$$0 \leq d_i - \nu_d \leq d_i - m_{\hat{d}} = \hat{d}_i - m_{\hat{d}}. \tag{20}$$

By the Lemma in Appendix A, when G contains no simple factors of type E_8, the right-most term is strictly negative when $d > 0$, giving a contradiction. For the E_8 case we have the stronger bound of Lemma 5^+ which applies to all simply laced types (see Lemma 6^+ below). Therefore, no such $d > 0$ terms arise, and the lemma is proved.

□

In the simply-laced case, we can say more.

Lemma 6^+. *If G is simply laced, then for* distinct $i_1, \ldots, i_l \in \{1, \ldots, r\}$, *we have* $P_{i_1} \star \cdots \star P_{i_l} = \prod_{k=1}^{l} P_{i_k}$. *That is, for these elements, the quantum and classical product are the same.* □

Proof. It suffices to show that for distinct $i_1, \ldots, i_l \in \{1, \ldots, r\}$, we have

$$\left(\prod_{k=1}^{l} q^{Q_{i_k} \partial_{Q_{i_k}}}\right) \widetilde{J} = \widetilde{\mathsf{T}}\left(\prod_{k=1}^{l} P_{i_k}\right).$$

This follows from the same argument as in the proof of Lemma 6. Indeed, the inequality in Equation (20) can be replaced by

$$0 \le \sum_{k=1}^{l} d_{i_k} - \nu_d \le \sum_{k=1}^{l} d_{i_k} - k_{\hat{d}} = -\left(\rho - \sum_{k=1}^{l} \varpi_{i_k}, \hat{d}\right) - \frac{(\hat{d}, \hat{d})}{2}.$$

The quantity $\left(\rho - \sum \varpi_{i_k}, \hat{d}\right)$ is nonnegative, and $\frac{(\hat{d},\hat{d})}{2}$ is strictly positive for $d \neq 0$, since $(\ ,\ )$ is an inner product. This contradicts the inequality, so no term with $d > 0$ occurs. □

4. Asymptotics of the fundamental solution T

We would like to establish a generalization of Lemma 4 (and Lemma 4^+ in simply-laced cases) to T_d by further exploring the properties of the zastava spaces. Alternatively, one may hope to derive such a generalization with the help of reconstruction theorems [24], [35]. The subtleties involved in either approach present formidable technical challenges.

We proceed differently. Lemmas 4 and 4^+ imply that J_d satisfies a *quadratic growth condition* in the sense introduced in Appendix B by H. Iritani. More precisely, for any smooth projective variety X, a linear operator $\mathsf{T} = \sum \mathsf{T}_d Q^d$ on $K_T(X)$ **satisfies the quadratic growth condition** if there is a positive-definite inner product $(\ ,\ )$ on $H_2(X)$, a linear functional m on $H_2(X)$, and a real constant c such that

$$\mathrm{ord}_{q=\infty} \mathsf{T}_d \ge \frac{(d,d)}{2} + m(d) + c$$

for all $d \in H_2(X)$. In the appendix, Iritani proves that the quadratic growth condition on the fundamental solution T is equivalent to the shift operators A_i being polynomials in the Novikov variables Q.

According to Kato [25, Corollary 3.3] (which uses our Lemma 6), for G/B the shift operators A_i are polynomials in Novikov variables Q. Applying Iritani's result (the Proposition of Appendix B), we obtain:

Lemma 7. *For G/P, the fundamental solution T satisfies the quadratic growth condition.*

Proof. By Iritani's Proposition and Kato's finiteness result for G/B [25, Corollary 3.3], the operator T for G/B satisfies the quadratic growth condition. Using the bounds of Lemma 5, the quadratic growth condition for G/P follows. □

Applying the Proposition of Appendix B again, it follows that the shift operators A_i for G/P are also polynomials in Q. We give a direct argument for this last implication in the next section.

Arguing as in the proof of [24, Lemma 3.3], we have the following lemma, which will be used in Section 5.

Lemma 8. *Consider* $\mathsf{U} \in K_T(G/P)[q][[Q]]$ *such that* $\mathsf{T}(\mathsf{U}) = 0$ *at* $q = \infty$. *Then* $\mathsf{T}(\mathsf{U}) = 0$.

Proof. Write $\mathsf{M} := \mathsf{T}(\mathsf{U})$. Expanding $\mathsf{M} = \sum_d \mathsf{M}_d Q^d$, $\mathsf{T} = \sum_d \mathsf{T}_d Q^d$, and $\mathsf{U} = \sum_d \mathsf{U}_d Q^d$ as series in Q, we will show $\mathsf{M} = 0$ by induction with respect to a partial order on effective curve classes $d \in \check{\Lambda}_+$. In fact, we will show $\mathsf{U}_d = 0$ for all d.

As rational functions in q, the coefficients T_d and U_d have the following properties: $\mathsf{T}_0 = \mathrm{id}$; T_d has poles only at roots of unity, is regular at $q = 0$ and $q = \infty$, and vanishes at $q = \infty$ for $d > 0$; and U_d is a polynomial in q. Since $\mathsf{T}_0 = \mathrm{id}$, it follows that $\mathsf{U}_0 = 0$.

The product formula expands to give

$$\mathsf{M}_d = \mathsf{U}_d + \sum_{\substack{d'+d''=d \\ d',d''>0}} \mathsf{T}_{d'} \mathsf{U}_{d''},$$

using $\mathsf{T}_d(\mathsf{U}_0) = \mathsf{T}_d(0) = 0$. By induction, the indexed sum is zero (since all lower terms $\mathsf{U}_{d''} = 0$), i.e., $\mathsf{M}_d = \mathsf{U}_d$. Since M_d vanishes at $q = \infty$ for all d, but U_d is a polynomial in q, it follows that $\mathsf{U}_d = 0$ and $\mathsf{M} = 0$. ☐

5. FINITENESS

We will deduce our main finiteness theorem from the following statement for products of the line bundle classes P_i. This argument originally appeared in our preprint [2, Proposition 5].

Proposition 9. *For any indices* $i_1, \dots, i_l$, *the (small) quantum product* $P_{i_1} \star \cdots \star P_{i_l}$ *is a finite linear combination of elements of* $K_T(G/P)$ *whose coefficients are polynomials in* $Q_1, \dots, Q_r$.

The statement is similar to the "only if" direction of Iritani's Proposition in Appendix B, but phrased differently. In our context, because of Lemma 6, polynomiality of $\mathsf{A}_{i,\mathrm{com}}$ is equivalent to that of quantum multiplication by P_i.

In proving Proposition 9, we will extend scalars from $R(T)$ to $F(T)$, and choose an $F(T)$-basis $\Phi_w = P^{\lambda(w)}$ of line bundles, for some $\lambda(w) \in \Lambda$. (By Lemma 1, $F(T) \otimes_{R(T)} K_T(G/P)$ is generated by line bundles over $F(T)$, so such a monomial basis exists.) This extension of scalars is harmless, for the following reason. A priori, we know the quantum product $P_{i_1} \star \cdots \star P_{i_l}$ lies in $K_T(G/P)[[Q]]$. The argument we give below shows that it lies in $(F(T) \otimes_{R(T)} K_T(G/P))[Q]$. This proves the claim, because the intersection of the submodules $K_T(G/P)[[Q]]$ and $(F(T) \otimes_{R(T)} K_T(G/P))[Q]$ inside $(F(T) \otimes_{R(T)} K_T(G/P))[[Q]]$ is precisely $K_T(G/P)[Q]$.

Proof. From Equation (10), for $I = (i_1, \dots, i_l)$ we have

$$\left(P^{\log Q/\log q}\right)^{-1} \prod_{k=1}^{l} q^{Q_{i_k}\partial_{Q_{i_k}}} \widetilde{J} = \mathsf{T}(a_I), \tag{21}$$

where $a_I := \prod_{i \in I} \mathsf{A}_i q^{Q_i \partial_{Q_i}}(1) \in F(T)[q][[Q]]$ by [24, Proposition 2.10]. This can be rewritten as in Equation (11), as

$$\prod_{k=1}^{l} P_{i_k} \left(\sum_{d \geq 0} q^{\sum_{k=1}^{l} d_{i_k}} J_d Q^d \right) = \mathsf{T}(a_I) = a_I^{(0)} + \sum_{d>0} a_I^{(d)} Q^d + \dots \tag{22}$$

where the omitted terms vanish at $q = \infty$ (since $\mathsf{T} = \mathrm{id} + O(q^{-1})$).

By Lemma 6, the operator $\mathsf{A}_{i,\mathrm{com}}$ is the operator of quantum multiplication by P_i. Along with Equation (17) for $I = (i_1, \dots, i_l)$, we obtain

$$P_{i_1} \star \cdots \star P_{i_l} = \prod_{k=1}^{l} \mathsf{A}_{i_k,\mathrm{com}}(1) = a_I|_{q=1}. \tag{23}$$

Our goal is to show that $a_I|_{q=1}$ is a polynomial in Q.

As in the proof of Lemma 6, we begin by showing that only finitely many Q^d appear in the $q^{\geq 0}$ coefficients of the left-hand side of Equation (22).

Note that the first term of the left hand side gives $\prod_{k=1}^{l} P_{i_k}$. Suppose a $d > 0$ term contributes to the $q^{\geq 0}$ coefficients on the left-hand side of Equation (22), i.e. suppose that $q^{\sum_{k=1}^{l} d_{i_k}} J_d$ has non-positive order at $q = \infty$. Then applying Lemma 5 gives

$$0 \leq \sum_{k=1}^{l} d_{i_k} - \nu_d \leq \sum_{k=1}^{l} d_{i_k} - m_{\hat{d}} = \sum_{k=1}^{l} \hat{d}_{i_k} - r(\hat{d}) - \frac{(\hat{d}, \hat{d})}{2}. \tag{24}$$

Here, as in the proof of Lemma 6, $\hat{d}$ is a lift of $d = d_P$ so that $m_{\hat{d}} = \min_{d_B^{\min} \leq d_B^+ \leq d_B} \{m_{d_B^+}\}$. So $\hat{d}_i = d_i$ for $i \notin I_P$.

There are finitely many possibilities for d which satisfy the inequality (24). Indeed, the quadratic form $(\ ,\ )$ is positive definite, so level sets of

$$\left(\sum_{k=1}^{l} \hat{d}_{i_k} - r(\hat{xd}) \right) - \frac{(\hat{d}, \hat{d})}{2}$$

(as a function of $\hat{d}$) are ellipsoids in the vector space $\check{\Lambda} \otimes \mathbb{R}$. It follows that the set

$$\left\{ d = (d_j)_{j \notin I_P} \middle| \left(\sum_{k=1}^{l} \hat{d}_{i_k} - r(\hat{d}) \right) - \frac{(\hat{d}, \hat{d})}{2} \geq 0 \right\}$$

is a bounded subset of $\check{\Lambda}^P \otimes \mathbb{R}$, so it can contain at most finitely many lattice points $d \in \check{\Lambda}_+^P$. Therefore the left hand side of Equation (22) (and hence of Equation (21)) has finitely many $q^{\geq 0}$ terms.

Since $\mathsf{T} = \mathrm{id} + O(q^{-1})$, we have

$$q^n Q^{d'} \mathsf{T}(\Phi_w) = q^n Q^{d'} \Phi_w + (\text{terms involving } q^{n'} \text{ for } n' < n).$$

In other words, the expansion of $q^n Q^{d'} \mathsf{T}(\Phi_w)$ has a unique term with maximal power of q. Ordering the finitely many terms of the left hand side of Equation (22) according to the exponents of q, we may therefore use the elements

$$q^n Q^{d'} \mathsf{T}(\Phi_w), \qquad \text{for } n \in \mathbb{Z}_{\geq 0},\ d' \in \check{\Lambda}_+^P,\ w \in W^P,$$

to inductively remove the $q^{\geq 0}$ terms.

By Lemma 7, the operator T satisfies the quadratic growth condition; it follows that for fixed n and w, the element $q^n Q^{d'} \mathsf{T}(\Phi_w)$ has only finitely many $q^{\geq 0}$ terms (essentially by repeating the argument given in the first part of this proof). So the inductive removal of $q^{\geq 0}$ terms ends after finitely many steps.[4] This means we can find *polynomials* $f_w \in F(T)[q, Q]$ so that the expression

$$\mathsf{M} := \prod_{k=1}^{l} P_{i_k} \left(\sum_{d \geq 0} q^{\sum_{k=1}^{l} d_{i_k}} J_d Q^d \right) - \sum_w \mathsf{T}(f_w \Phi_w) \tag{25}$$

vanishes at $q = +\infty$.

From Equations 10 and 11, this is also equal to

$$\begin{aligned} \mathsf{M} &= \left(P^{\log Q/\log q}\right)^{-1} \left(\prod_{k=1}^{l} q^{Q_{i_k} \partial_{Q_{i_k}}} \widetilde{J} - \sum_w \widetilde{\mathsf{T}}(f_w \Phi_w) \right) \\ &= \mathsf{T} \left(\left(\prod \mathsf{A}_i \, q^{Q_{i_k} \partial_{Q_{i_k}}} \right) (1) - \sum_w f_w \Phi_w \right) \\ &=: \mathsf{T}(\mathsf{U}). \end{aligned}$$

By Lemma 8—which holds without change after the extension of scalars from $R(T)$ to $F(T)$—we conclude that $\mathsf{M} = 0$. □

In particular, the proof of Proposition 9 gives the following refinement of Equation (21):

$$\prod_{k=1}^{l} q^{Q_{i_k} \partial_{Q_{i_k}}} \widetilde{J} = \sum_w \widetilde{\mathsf{T}}(f_w \Phi_w)$$

for *polynomials* $f_w \in R(T)[q][Q]$, giving

$$\prod_{k=1}^{l} P_{i_1} \star \cdots \star P_{i_l} = \sum_w f_w \Phi_w.$$

We now turn to our main theorem. We fix an $R(T)$-basis $\{\Phi_w\}$ for $K_T(G/P)$, and recycle the notation to write $\Phi_w = \Phi_w \otimes 1$ for the corresponding $R(T)[[Q]]$-basis of $QK_T(G/P) := K_T(G/P) \otimes \mathbb{Z}[[Q]]$.

[4]We stress that this step is *the only* part of our approach that uses bounds for T.

Theorem 10. *The structure constants of $QK_T(G/P)$ with respect to the basis $\{\Phi_w\}$ are polynomials: they lie in the polynomial subring $R(T)[Q]$ of $R(T)[[Q]]$.*

In particular, taking Φ_w to be a Schubert basis (of structure sheaves, canonical sheaves, or dual structure sheaves), we see that the quantum product of Schubert classes in $QK_T(G/P)$ is finite.

Proof. We begin by extending scalars from $R(T)$ to the fraction field $F(T)$ of $R(T)$, as in Proposition 9; the structure constants are automatically in $R(T)[[Q]]$, so to prove they lie in $R(T)[Q]$, it is enough to show they lie in $F(T)[Q]$.

The assignment $P_{i_1}P_{i_2}\cdots P_{i_k} \mapsto P_{i_1}\star P_{i_2}\star\cdots\star P_{i_k}$ defines a ring homomorphism

$$F(T)[P_1,\dots,P_r;Q_1,\dots,Q_r] \to F(T)\otimes_{R(T)} QK_T(G/P); \tag{26}$$

let the kernel be I. The resulting embedding of rings

$$F(T)[P_1,\dots,P_r;Q_1,\dots,Q_r]/I \hookrightarrow F(T)\otimes_{R(T)} QK_T(G/P)$$

corresponds to the natural embedding of modules

$$F(T)\otimes_{R(T)} K_T(G/P)\otimes\mathbb{Z}[Q_1,\dots,Q_r] \hookrightarrow F(T)\otimes_{R(T)} K_T(G/P)\otimes\mathbb{Z}[[Q_1,\dots,Q_r]].$$

It follows from Lemma 1 that each element Φ_w of the $R(T)$-basis for $K_T(G/P)$ can be written as a polynomial in P_i with coefficients in $F(T)$. Therefore, each element Φ_w of the corresponding $R(T)[[Q]]$-basis for $QK_T(G/P)$ can be represented as a polynomial $\varphi_w = \varphi_w(P,Q)$ in $F(T)[P_1,\dots,P_r][Q]$

The product of basis elements $\Phi_u\star\Phi_v$ in $QK_T(G/P)$ is given by a product $\varphi_u\,\varphi_v$ of polynomials in P and Q, and by Proposition 9, this product is a finite linear combination of classes in $F(T)\otimes_{R(T)} K_T(G/P)$ with coefficients in $\mathbb{Z}[Q]$. □

APPENDIX A. AN INEQUALITY IN THE COROOT LATTICE

Consider a root system (of finite type) in a real vector space V, with simple roots $\alpha_1,\dots,\alpha_r$ and associated reflection group W. Let $d=\sum_j d_j\alpha_j$ be an element of the root lattice, so the coefficients d_j are integers. Let $(\ ,\ )$ be the W-invariant bilinear form on V, normalized so that $(\alpha_j,\alpha_j)=2$ for short roots. Finally, let

$$r(d)=\#\{j \mid d_j\neq 0\}.$$

The purpose of this appendix is to prove a simple inequality.

Lemma. *Assume that the root system contains no factors of type* E_8. For any $i\in\{1,\dots,r\}$, we have

$$\frac{(d,d)}{2}+r(d)\geq d_i,$$

with equality if and only if $d=0$.

Proof. We may assume $r(d) = r$, i.e., d has full support, since otherwise the problem reduces to a root subsystem.

Let us introduce a new variable z, and consider the quadratic form

$$Q(d_1, \ldots, d_r, z) = \frac{(d, d)}{2} - d_i z + r z^2.$$

We will show that Q is positive definite. The lemma follows, by evaluating at $z = 1$.

Let us write A_Q for the symmetric matrix corresponding to Q, A_R for the matrix corresponding to $\frac{1}{2}(\ ,\ )$, and $A_{R(i)}$ for the matrix of the subsystem obtained by removing α_i. By reordering the roots as needed, we can assume A_R and $A_{R(i)}$ are principal submatrices of A_Q, so $2A_Q$ has the form

$$2A_Q = \left(\begin{array}{ccc|c} & & & 0 \\ & 2A_R & & \vdots \\ & & & -1 \\ \hline 0 & \cdots & -1 & 2r \end{array} \right)$$

We see

$$\det(2A_Q) = 2r\ \det(2A_R) - \det(2A_{R(i)}).$$

To prove that Q is positive definite, it suffices to check this determinant is positive, since we already know A_R is positive definite. This is easily done with a case-by-case check, using the data in Table 1. (Cf. [22, §2.4], noting that our matrices are multiplied by factors corresponding to long roots.) □

R	A_n	B_n	C_n	D_n	E_6	E_7	F_4	G_2
$\det(2A_R)$	$n+1$	2^n	4	4	3	2	4	3

TABLE 1. Determinants for root systems

Remark. In type E_8, if i corresponds to the vertex of degree 3 (the "fork") in the Dynkin diagram, then the quadratic form Q is not positive definite: in fact, the determinant $\det(2A_Q)$ is negative in this case.

APPENDIX B. FINITENESS AND QUADRATIC GROWTH IN QUANTUM K-THEORY

by Hiroshi Iritani[5]

We show that a quadratic growth condition for the zero orders of the fundamental solution T at $q = \infty$ is equivalent to the finiteness of the q-shift connection A associated with nef classes.

Let X be a smooth projective variety. Let $K(X)$ be the topological K-group with complex coefficients. We fix a basis $\{\Phi_\alpha\}$ of $K(X)$. Let g denote the pairing on $K(X)$ given by $g(E, F) = \chi(E \otimes F)$. Let $\{\Phi^\alpha\}$ denote the dual basis with respect to the pairing g. Let T denote the fundamental solution of the quantum difference equation, defined by

$$\mathsf{T}(\Phi_\alpha) = \Phi_\alpha + \sum_{\substack{d \in \mathrm{Eff}(X) \\ d \neq 0}} \sum_\beta \left\langle \Phi_\alpha, \frac{\Phi_\beta}{1 - qL} \right\rangle_{0,2,d} Q^d \Phi^\beta.$$

where $\mathrm{Eff}(X) \subset H_2(X, \mathbb{Z})$ denotes the monoid generated by effective curves. We write $\mathsf{T} = \sum_{d \in \mathrm{Eff}(X)} \mathsf{T}_d Q^d$ with $\mathsf{T}_d \in \mathrm{End}(K(X))$. We say that T *satisfies the quadratic growth condition* when the following holds:

> There exist a positive-definite inner product $(\cdot, \cdot)$ on $H_2(X)$, $m \in H^2(X)$ and a constant $c \in \mathbb{R}$ such that we have
>
> $$\mathrm{ord}_{q=\infty} \mathsf{T}_d \geq \frac{1}{2}(d, d) + m \cdot d + c \tag{B.1}$$
>
> for all $d \in H_2(X)$, where $\mathrm{ord}_{q=\infty}$ is the order of zero at $q = \infty$.

For a class $P \in K(X)$ of a line bundle, we write $p = -c_1(P) \in H^2(X)$ for the *negative* of the first Chern class. For $p \in H^2(X)$, let $q^{pQ\partial_Q}$ denote the operator acting on power series in Q as

$$q^{pQ\partial_Q} \left(\sum_{d \in H_2(X)} c_d Q^d \right) = \sum_{d \in H_2(X)} c_d q^{p \cdot d} Q^d.$$

The q-shift connection A associated with P (or with $p = -c_1(P)$) is the operator

$$\mathsf{A} = \mathsf{T}^{-1} P q^{pQ\partial_Q}(\mathsf{T})$$

where P acts on $K(X)$ by the (classical) tensor product. The nontrivial fact is that A lies in the ring $\mathrm{End}(K(X)) \otimes \mathbb{C}[q, q^{-1}][\![Q]\!]$, i.e. it is a Laurent polynomial in q.

Proposition. *The fundamental solution* T *satisfies the quadratic growth condition* (B.1) *if and only if the difference connections* A *associated with nef classes* $p = -c_1(P)$ *are polynomials in* Q.

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Proof. The 'only if' statement was (essentially) proved by Anderson-Chen-Tseng [2, Proposition 5] (see also Proposition 9 above) although it was not phrased in this way. We give another proof for the convenience of the reader. We expand $\mathsf{T}^{-1} = (1 + \sum_{d\neq 0} \mathsf{T}_d Q^d)^{-1} = \sum_d \mathsf{S}_d Q^d$. Then:

$$\mathsf{S}_d = \sum_{k\geq 1} \sum_{\substack{d(1)+\cdots+d(k)=d,\\ d(j)\in \mathrm{Eff}(X)\setminus\{0\}}} (-1)^k \mathsf{T}_{d(1)} \cdots \mathsf{T}_{d(k)}$$

for $d \neq 0$.

We claim that $\mathrm{ord}_{q=\infty} \mathsf{S}_d \to \infty$ as $|d| := \sqrt{(d,d)} \to \infty$. By the quadratic growth condition (B.1) and the fact that $\mathrm{ord}_{q=\infty} \mathsf{T}_d \geq 1$ for $d \neq 0$, when $d = d(1) + \cdots + d(k)$ with $d(j) \in \mathrm{Eff}(X) \setminus \{0\}$, we have

$$\mathrm{ord}_{q=\infty}(\mathsf{T}_{d(1)} \cdots \mathsf{T}_{d(k)}) \geq \max(k, f(d(1)) + \cdots + f(d(k))) \tag{B.2}$$

where $f(d) := \frac{1}{2}(d,d) + m\cdot d + c$. Since $|d| \leq |d(1)| + \cdots + |d(k)|$, there exists i such that $|d(i)| \geq |d|/k$. Therefore if $k \leq |d|^{\frac{1}{3}}$, then

$$\begin{aligned} f(d(1)) + \cdots + f(d(k)) &= \frac{1}{2}\left(\sum_{i=1}^{k} (d(i), d(i))\right) + m\cdot d + ck \\ &\geq \frac{1}{2}\frac{|d|^2}{k^2} - |m||d| - |c|k \\ &\geq \frac{1}{2}|d|^{\frac{4}{3}} - |m||d| - |c||d|^{\frac{1}{3}} \end{aligned}$$

Hence by (B.2),

$$\mathrm{ord}_{q=\infty}(\mathsf{T}_{d(1)} \cdots \mathsf{T}_{d(k)}) \geq \min\left(|d|^{\frac{1}{3}}, \frac{1}{2}|d|^{\frac{4}{3}} - |m||d| - |c||d|^{\frac{1}{3}}\right)$$

and the right-hand side diverges as $|d| \to \infty$. This proves the claim.

Let A be the q-shift operator associated with a nef class $p = -c_1(P)$. Writing $\mathsf{A} = \sum_d \mathsf{A}_d Q^d$, we have

$$\mathsf{A}_d = \sum_{d'+d''=d} \mathsf{S}_{d'} P q^{p\cdot d''} \mathsf{T}_{d''}.$$

Since p is nef, A is regular at $q = 0$ (see [24, Proposition 2.10]). On the other hand, using the quadratic growth condition (B.1) again, we have

$$\mathrm{ord}_{q=\infty} \mathsf{A}_d \geq \min_{d'+d''=d}(\mathrm{ord}_{q=\infty} \mathsf{S}_{d'} + f(d'') - p\cdot d'').$$

The right-hand side is positive for a sufficiently large $|d|$. In fact, both $N' = \{d' \in \mathrm{Eff}(X) : \mathrm{ord}_{q=\infty} \mathsf{S}_{d'} < 0\}$ and $N'' = \{d'' \in \mathrm{Eff}(X) : f(d'') - p\cdot d'' < 0\}$ are finite sets; when $d' \in N'$ and $d' + d'' = d$, we have $f(d'') - p\cdot d'' \to \infty$ as $|d| \to \infty$; similarly, when $d'' \in N''$ and $d' + d'' = d$, we have $\mathrm{ord}_{q=\infty} \mathsf{S}_{d'} \to \infty$ as $|d| \to \infty$. Therefore A_d is regular at $q = 0$ and $\mathrm{ord}_{q=\infty} \mathsf{A}_d > 0$ for sufficiently large $|d|$. This implies that $\mathsf{A}_d = 0$ for sufficiently large $|d|$, i.e. A is a polynomial in Q.

Next we show the 'if' statement. Suppose that all q-shift connections A associated with nef classes $p = -c_1(P)$ are polynomials in Q. Choose line bundles $P_1, \dots, P_k$ such that $p_i = -c_1(P_i)$ is nef and that $p_1, \dots, p_k$ form a basis of $H^2(X, \mathbb{R})$. Let $\mathsf{A}^{(i)}$ be the q-shift connection associated with P_i. By assumption, there exists a finite set $F \subset \mathrm{Eff}(X) \setminus \{0\}$ of degrees such that $\mathsf{A}^{(i)}$ is expanded in the form:

$$\mathsf{A}^{(i)} = P_i + \sum_{d \in F} \mathsf{A}^{(i)}_d Q^d.$$

The fundamental solution T satisfies the q-difference equation $P_i q^{p_i Q \frac{\partial}{\partial Q}} \mathsf{T} = \mathsf{T}\mathsf{A}^{(i)}$, and therefore we have

$$P_i q^{p_i \cdot d} \mathsf{T}_d = \mathsf{T}_d P_i + \sum_{d' \in F} \mathsf{T}_{d-d'} \mathsf{A}^{(i)}_{d'}. \tag{B.3}$$

Suppose $p_i \cdot d > 0$. Then we have

$$\mathrm{ord}_{q=\infty} \mathsf{T}_d \geq p_i \cdot d + \min_{d' \in F} (\mathrm{ord}_{q=\infty} \mathsf{T}_{d-d'}) + C$$

where $C := \min_{1 \leq i \leq k, d' \in F} (\mathrm{ord}_{q=\infty} \mathsf{A}^{(i)}_{d'})$. Note that the first term in the right-hand side of (B.3) does not contribute to the vanishing order of T_d at $q = \infty$ because $p_i \cdot d > 0$. Since this holds for all i with $p_i \cdot d > 0$, and there exists at least one i with $p_i \cdot d > 0$ when $d \in \mathrm{Eff}(X) \setminus \{0\}$ (note that $p_i \cdot d \geq 0$ since p_i is nef), we have

$$\mathrm{ord}_{q=\infty} \mathsf{T}_d \geq \max_{1 \leq i \leq k} (p_i \cdot d) + \min_{d' \in F} (\mathrm{ord}_{q=\infty} \mathsf{T}_{d-d'}) + C \tag{B.4}$$

for all $d \in \mathrm{Eff}(X) \setminus \{0\}$. Introduce the norm $\|d\| := \sqrt{\sum_{i=1}^k (p_i \cdot d)^2}$ and set $B := \max_{d \in F} \|d\|$. Define the positive-definite inner product $(\cdot, \cdot)$ on $H_2(X)$ by

$$(d', d'') = \frac{1}{\sqrt{k} B} \sum_{i=1}^k (p_i \cdot d')(p_i \cdot d'').$$

Choose a class $m \in H^2(X)$ such that $m \cdot d \leq C$ for all $d \in F$. This is possible since F is a finite set contained in $\mathrm{Eff}(X) \setminus \{0\}$. We claim that

$$\mathrm{ord}_{q=\infty} \mathsf{T}_d \geq \frac{1}{2}(d, d) + m \cdot d. \tag{B.5}$$

This is true for $d = 0$. We introduce a partial order $\prec$ in $\mathrm{Eff}(X)$ so that $d \prec d'$ if and only if $d' - d \in \mathrm{Eff}(X)$. Since every infinite descending chain $d(1) \succ d(2) \succ d(3) \succ \cdots$ in $\mathrm{Eff}(X)$ stabilizes, the induction argument works for this order. Suppose that $d_* \in \mathrm{Eff}(X) \setminus \{0\}$ and that (B.5) holds for all $d \in \mathrm{Eff}(X)$ with $d \prec d_*$. Using

(B.4) and the induction hypothesis, we have

$$\begin{aligned}
\operatorname{ord}_{q=\infty} \mathsf{T}_{d_*} &\geq \max_{1\leq i\leq k} (p_i \cdot d_*) + \min_{d'\in F} \left(\frac{1}{2}(d_* - d', d_* - d') + m \cdot (d_* - d')\right) + C \\
&\geq \frac{1}{2}(d_*, d_*) + m \cdot d_* + \max_{1\leq i\leq k} (p_i \cdot d_*) - \max_{d'\in F}(d_*, d') - \max_{d'\in F}(m \cdot d') + C \\
&\geq \frac{1}{2}(d_*, d_*) + m \cdot d_* + \frac{1}{\sqrt{k}}\|d_*\| - \sqrt{(d_*, d_*)} \max_{d'\in F} \sqrt{(d', d')} \\
&\geq \frac{1}{2}(d_*, d_*) + m \cdot d_* + \frac{1}{\sqrt{k}}\|d_*\| - \frac{1}{\sqrt{k}B}\|d_*\| \max_{d'\in F} \|d'\| \\
&\geq \frac{1}{2}(d_*, d_*) + m \cdot d_*.
\end{aligned}$$

In the above computation, we used $\|d_*\| \leq \sqrt{k} \max_{1\leq i\leq k}(p_i \cdot d_*)$. Hence the estimate (B.5) holds for d_*. The proposition is proved. □

Remark. The Proposition holds also for the equivariant quantum K-theory. The proof works verbatim.

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