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MOTIVIC CLASSES OF DEGENERACY LOCI AND POINTED BRILL-NOETHER VARIETIES

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Abstract. Motivic Chern and Hirzebruch classes are polynomials with K-theory and homology classes as coefficients, which specialize to Chern-Schwartz-MacPherson classes, K-theory classes, and Cappell-Shaneson L-classes. We provide formulas to compute the motivic Chern and Hirzebruch classes of Grassmannian and vexillary degeneracy loci. We apply our results to obtain the Hirzebruch χ_y-genus of classical and one-pointed Brill-Noether varieties, and therefore their topological Euler characteristic, holomorphic Euler characteristic, and signature.

The study of motivic Chern classes unifies several theories of characteristic classes of singular varieties [10]. The motivic Chern and Hirzebruch classes are polynomials in a formal variable y; the motivic Hirzebruch class T_y specializes to the Chern-Schwartz-MacPherson (CSM) class for $y = -1$, the K-theoretic Todd class for $y = 0$, and the Cappell-Shaneson L-class for $y = 1$. The top degree term of T_y of a compact variety gives the Hirzebruch χ_y-genus which specializes to the topological Euler characteristic for $y = -1$, the holomorphic Euler characteristic for $y = 0$, and the signature for $y = 1$.

In this article, we compute the motivic Chern and Hirzebruch classes of Grassmannian and vexillary degeneracy loci in type A. In particular, our results give formulas for their CSM classes and L-classes. Several invariants of these degeneracy loci have been computed, e.g., a determinantal formula for their classes in cohomology [19, 8], in K-theory [27, 5], and in algebraic cobordism [28]. In the important special case of a degeneracy locus of a single map between vector bundles, the CSM class was computed by Parusiński-Pragacz [34]. However, formulas for their L-classes and for CSM classes of more general degeneracy loci were not known. CSM classes and motivic Chern classes have recently been studied for Schubert varieties and Schubert cells in flag manifolds [1, 29, 2, 3, 4] and matrix Schubert cells [17, 37, 18]. Hirzebruch χ_y-genera have also been computed in other instances, e.g., for Hilbert schemes of points [24, 11] and for singular toric varieties [32].

We consider maps of vector bundles over a smooth algebraic variety X:

$$E_p \xrightarrow{\varphi} F_{q_1} \twoheadrightarrow F_{q_2} \cdots \twoheadrightarrow F_{q_t}$$

with $\operatorname{rank}(E_p) = p$ and $\operatorname{rank}(F_{q_i}) = q_i$. The *Grassmannian degeneracy locus* corresponding to the partition $\boldsymbol{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_t \geq 0)$ with $\lambda_i := q_i - p + i$ is defined as

$$W_{\boldsymbol{\lambda}} := \{x \in X \,:\, \dim \ker (E_p \to F_{q_i})\,|_x \geq i\}\,.$$

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More generally, we will consider maps of vector bundles

$$E_{p_1} \hookrightarrow E_{p_2} \cdots \hookrightarrow E_{p_t} \xrightarrow{\varphi} F_{q_1} \twoheadrightarrow F_{q_2} \cdots \twoheadrightarrow F_{q_t} \tag{1}$$

over a smooth algebraic variety X, with $\operatorname{rank}(E_{p_i}) = p_i$ and $\operatorname{rank}(F_{q_i}) = q_i$. Note that $0 < p_1 \leq \cdots \leq p_t$ and $q_1 \geq \cdots \geq q_t > 0$. Given a (weakly) increasing sequence $\boldsymbol{k} := (k_1, \ldots, k_t)$ of positive integers, the *vexillary degeneracy locus* corresponding to the triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ is defined as

$$W_{\boldsymbol{\tau}} := \{x \in X \,:\, \dim \ker (E_{p_i} \to F_{q_i})\,|_x \geq k_i\}\,.$$

This nomenclature arises because the rank conditions can be described by the *vexillary permutations* from [31] (i.e., permutations avoiding the pattern 2 1 4 3), see [7, §1]. The Grassmannian case is recovered when $p_1 = \cdots = p_t$ and $k_i = i$ for each i.

Our main result gives formulas to compute the motivic Chern and Hirzebruch classes of vexillary degeneracy loci. We proceed in two steps.

First, we relate the motivic Hirzebruch classes of a vexillary degeneracy locus $W_{\boldsymbol{\tau}}$ and a certain resolution of $W_{\boldsymbol{\tau}}$. As in Kempf-Laksov [30], $W_{\boldsymbol{\tau}}$ is resolved by $\phi\colon \widetilde{\Omega}_{\boldsymbol{\tau}} \to W_{\boldsymbol{\tau}}$ where $\widetilde{\Omega}_{\boldsymbol{\tau}}$ is the variety parametrizing complete flags of sub-bundles $V_1 \subseteq \cdots \subseteq V_{k_t}$ such that $\operatorname{rank}(V_i) = i$ and $V_{k_i} \subseteq \ker(E_{p_i} \to F_{q_i})$ for each i, see §3. (In §6 we will also study a partial resolution $\widetilde{\Omega} \to \Omega \to W$, hence the use of the tilde here.) Let $\iota\colon W_{\boldsymbol{\tau}} \hookrightarrow X$ denote the inclusion. We carry out an explicit computation of the class $(\iota\phi)_* T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right)$:

Theorem 1. *For a triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ and with assumptions as in §1.4, the class $(\iota\phi)_* T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right)$ is computed by a universal operator applied to $[W_{\boldsymbol{\tau}}] \cap T_y(X)$, where $[W_{\boldsymbol{\tau}}]$ is the determinantal formula for the class of $W_{\boldsymbol{\tau}}$ in $A^*(X)$ (explicitly, Theorem 3.1).*

Second, since the fibers of ϕ are not constant, in order to compare the motivic Hirzebruch classes of $W_{\boldsymbol{\tau}}$ and $\widetilde{\Omega}_{\boldsymbol{\tau}}$, we proceed to find the stratification of $W_{\boldsymbol{\tau}}$ into locally closed strata on which ϕ is locally trivial.

As reviewed in §1.1.2 after [8], a triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ can be inflated to a triple $\boldsymbol{\tau}' = (\boldsymbol{k}', \boldsymbol{p}', \boldsymbol{q}')$ by inflating the sequences $\boldsymbol{k}$, $\boldsymbol{p}$, and $\boldsymbol{q}$ of length t to sequences $\boldsymbol{k}'$, $\boldsymbol{p}'$, and $\boldsymbol{q}'$ of length k_t with $\boldsymbol{k}' = (1, 2, \ldots, k_t)$ such that

$$q'_1 - p'_1 + 1 \;>\; q'_2 - p'_2 + 2 \;>\; \cdots \;>\; q'_{k_t} - p'_{k_t} + k_t \;> 0$$

and $W_{\boldsymbol{\tau}} = W_{\boldsymbol{\tau}'}$. The locus $W_{\boldsymbol{\tau}}$ contains the loci $W_{\boldsymbol{\tau}^+}$ for $\boldsymbol{\tau}^+ = \left(\boldsymbol{k}^+, \boldsymbol{p}', \boldsymbol{q}'\right)$ with $\boldsymbol{k}^+ \geq \boldsymbol{k}'$ in componentwise order. For degree reasons, there are only finitely many such sub-loci $W_{\boldsymbol{\tau}^+} \subseteq W_{\boldsymbol{\tau}}$. The map ϕ is locally trivial precisely on the locally closed strata $W^\circ_{\boldsymbol{\tau}^+} \subset W_{\boldsymbol{\tau}^+}$ (defined in (3) as expected) — see §5.1. We express the class $(\iota\phi)_* T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right)$ computed in Theorem 1 in terms of the motivic Hirzebruch classes of the strata of $W_{\boldsymbol{\tau}}$:

Theorem 2. *For a triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ and with assumptions as in §1.4, one has*

$$(\iota\phi)_* T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right) = \sum_{\boldsymbol{k}^+} (-y)^{|\boldsymbol{k}^+| - |\boldsymbol{k}'|}\; \iota_*\, T_y\left(W_{\boldsymbol{\tau}^+}\right)$$

where $\boldsymbol{\tau}^+ = \left(\boldsymbol{k}^+, \boldsymbol{p}', \boldsymbol{q}'\right)$ and the sum is over the set of weakly increasing sequences $\boldsymbol{k}^+ \geq \boldsymbol{k}' = (1, \ldots, k_t)$.

For instance, consider the triples $\boldsymbol{\tau}(i) := ((i),(3),(3))$, for $i = 1,2,3$, and the triple $\boldsymbol{\tau}(22) := ((2,2),(2,3),(3,3))$. The strata required for the locus $W_{\boldsymbol{\tau}(1)}$ are $W^\circ_{\boldsymbol{\tau}(i)}$ for $i = 1,2,3$. Since the triple $\boldsymbol{\tau}(2)$ is inflated to the triple $((1,2),(2,3),(3,3))$, the strata required for the locus $W_{\boldsymbol{\tau}(2)}$ are $W^\circ_{\boldsymbol{\tau}(2)}$, $W^\circ_{\boldsymbol{\tau}(3)}$, and $W^\circ_{\boldsymbol{\tau}(22)}$. Then Theorem 2 gives

$$\begin{aligned}
(\iota\phi)_*\, T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}(1)}\right) &= \iota_*\, T_y\left(W_{\boldsymbol{\tau}(1)}\right) - y\,\iota_*\, T_y\left(W_{\boldsymbol{\tau}(2)}\right) + y^2\,\iota_*\, T_y\left(W_{\boldsymbol{\tau}(3)}\right),\\
(\iota\phi)_*\, T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}(2)}\right) &= \iota_*\, T_y\left(W_{\boldsymbol{\tau}(2)}\right) - y\,\iota_*\, T_y\left(W_{\boldsymbol{\tau}(22)}\right) - y\,\iota_*\, T_y\left(W_{\boldsymbol{\tau}(3)}\right),\\
(\iota\phi)_*\, T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}(3)}\right) &= \iota_*\, T_y\left(W_{\boldsymbol{\tau}(3)}\right),\\
(\iota\phi)_*\, T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}(22)}\right) &= \iota_*\, T_y\left(W_{\boldsymbol{\tau}(22)}\right).
\end{aligned}$$

Finally, by the inclusion-exclusion principle, applying Theorem 2 to the (closure of the) strata of $W_{\boldsymbol{\tau}}$, and then to the strata of the strata, and so on, one can express the motivic Hirzebruch class of $W_{\boldsymbol{\tau}}$ in terms of classes $(\iota\phi)_*\, T_y\left(\widetilde{\Omega}_{\boldsymbol{\sigma}}\right)$ corresponding to a subset of the finitely many vexillary degeneracy sub-loci $W_{\boldsymbol{\sigma}} \subseteq W_{\boldsymbol{\tau}}$. Thus we have:

Theorem 3. *Combining Theorems 1 and 2 allows one to compute the motivic Hirzebruch class of $W_{\boldsymbol{\tau}}$ for any triple $\boldsymbol{\tau}$.*

As an example, for the triple $\boldsymbol{\tau}(1) := ((1),(3),(3))$ as above, solving for $\iota_*\, T_y\left(W_{\boldsymbol{\tau}(1)}\right)$ gives

$$\iota_*\, T_y\left(W_{\boldsymbol{\tau}(1)}\right) = (\iota\phi)_*\, T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}(1)}\right) + y\,(\iota\phi)_*\, T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}(2)}\right) + y^2\,(\iota\phi)_*\, T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}(22)}\right).$$

Theorem 1 can then be applied to compute the right-hand side as a polynomial in y with coefficients expressed in terms of the Chern classes of the given vector bundles.

Brill-Noether theory. In §7 we apply our results to classical and pointed Brill-Noether varieties. Brill-Noether theory studies the geometry of line bundles and linear series on algebraic curves. For a smooth algebraic curve C, the *classical Brill-Noether variety* $W^r_d(C)$ parametrizes line bundles of degree d on C having at least $r+1$ independent global sections [9]. More generally, the *pointed Brill-Noether variety* $W^{\boldsymbol{a}}_d(C,P)$ parametrizes line bundles of degree d on C having at least $r+1$ independent global sections with vanishing orders at the point P at least equal to $\boldsymbol{a} = (0 \le a_0 < \cdots < a_r \le d)$. In §7.3, we compute the motivic Hirzebruch class of $W^r_d(C)$ and $W^{\boldsymbol{a}}_d(C,P)$ for a general (C,P), as these are examples of Grassmannian degeneracy loci. This extends the study of the CSM class of the classical Brill-Noether varieties $W^r_d(C)$ treated by Parusiński-Pragacz [34].

Similarly, we compute the motivic Hirzebruch class of the *Brill-Noether variety* $G^r_d(C)$ parametrizing linear series on C of degree d and projective dimension r, and its pointed counterpart $G^{\boldsymbol{a}}_d(C,P)$ parametrizing linear series on C of degree d with prescribed vanishing $\boldsymbol{a}$ at the point P. For these, we use the result presented more generally about a degeneracy locus $\Omega_{\boldsymbol{\lambda}}$ in §6.

For a smooth *Brill-Noether-Petri general* curve C of genus g, one has: (i) $G^r_d(C)$ is smooth and has dimension equal to $\rho(g,r,d) := g-(r+1)(g-d+r)$; (ii) $W^r_d(C)$ has dimension equal to $\rho(g,r,d)$, provided that $g-d+r \ge 0$; and (iii) when $g-d+r > 0$, the singular locus of $W^r_d(C)$ coincides with $W^{r+1}_d(C) \subset W^r_d(C)$ [9, pg. 214].

For instance, $W^r_d(C)$ is smooth when C is a smooth Brill-Noether-Petri general curve, $\rho(g,r,d) \leq 2$ and $g \geq 2$. Write $\lambda = g-d+r$, and let θ be the cohomology class of the theta divisor in $\mathrm{Pic}^d(C)$. In the surface case, we prove:

Corollary 4. *Fix $g \geq 2$ and r, d such that $\rho(g,r,d) = 2$. For a Brill-Noether-Petri general smooth curve C of genus g, the motivic Hirzebruch class of the surface $W^r_d(C)$ is*

$$T_y\left(W^r_d(C)\right) = \left(1 + \frac{\lambda(r+3)(y-1)}{2(\lambda+2)}\,\theta + \frac{\lambda(r+1)\left(\lambda(r+1)(y-1)^2 - 2y\right)}{2(\lambda+r)(\lambda+r+2)}\theta^2\right) \cdot \theta^{g-2} \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}$$

in $H^*\left(\mathrm{Pic}^d(C)\right)[y]$. Since the top degree term of T_y gives the Hirzebruch χ_y-genus, one has

$$\chi_y\left(W^r_d(C)\right) = g! \frac{\lambda(r+1)\left(\lambda(r+1)(y-1)^2 - 2y\right)}{2(\lambda+r)(\lambda+r+2)} \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}.$$

Moreover, with the same hypotheses, one has $\chi_y\left(G^r_d(C)\right) = \chi_y\left(W^r_d(C)\right)$.

The formula for χ_y in Corollary 4 recovers the topological and holomorphic Euler characteristics for $y=-1$ and $y=0$ (known from [34] and [6], respectively), and gives a new result on the *signature* of the surface $W^r_d(C)$ for $y=1$:

$$\sigma\left(W^r_d(C)\right) = g! \frac{2-g}{(g-d+2r)(g-d+2r+2)} \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}$$

and similarly for the *surface* $G^r_d(C)$, since in this case $\sigma\left(G^r_d(C)\right) = \sigma\left(W^r_d(C)\right)$.

We briefly review the definition of the signature. For a compact oriented manifold X of real dimension $4k$, consider the non-degenerate symmetric bilinear form on the finite-dimensional vector space $H^{2k}(X,\mathbb{R})$ given by

$$\langle \alpha, \beta \rangle := \int_X \alpha \cup \beta, \qquad \text{for } \alpha, \beta \in H^{2k}(X,\mathbb{R}).$$

The signature $\sigma(X)$ of X is defined as the number of positive entries minus the number of negative entries in a diagonalized version of this form. When X is smooth, Hirzebruch's signature theorem expresses $\sigma(X)$ as a universal linear combination of the Pontrjagin numbers of the tangent bundle of X [25]. When X is possibly singular, $\sigma(X)$ is computed by the top degree term of the L-class of X [12, 10].

When $g=2$, then $W^r_d(C) = \mathrm{Pic}^d(C)$, thus indeed $\chi_y = 0$, since Abelian varieties have trivial tangent bundle. As a further check, σ and the topological Euler characteristic χ_{top} always have the same parity [35, Cor. 64], and this is indeed satisfied by the formulas resulting from Corollary 4. In fact, we observe the following, perhaps surprising, relations for the surface $W^r_d(C)$:

$$\begin{aligned} \frac{g-2}{2}\,\sigma\left(W^r_d(C)\right) &= -\chi_{\mathrm{hol}}\left(W^r_d(C)\right), \\ (2g-3)\,\sigma\left(W^r_d(C)\right) &= -\chi_{\mathrm{top}}\left(W^r_d(C)\right), \\ (g-2)\,\chi_{\mathrm{top}}\left(W^r_d(C)\right) &= (4g-6)\,\chi_{\mathrm{hol}}\left(W^r_d(C)\right). \end{aligned}$$

In §7.6 we also compute explicitly the motivic Hirzebruch class of pointed Brill-Noether surfaces parametrizing pencils (i.e., $r = 1$). Interestingly, we show that the motivic Hirzebruch classes of Grassmannian degeneracy loci corresponding to a partition $\boldsymbol{\lambda}$ do not specialize from the case $\lambda_i > \lambda_{i+1}$ to the case $\lambda_i = \lambda_{i+1}$, for some i (Remark 7.6). This is in contrast with the K-theory class (the case $y = 0$) of degeneracy loci with rank conditions imposed by arbitrary (not only vexillary) permutations, given by Grothendieck polynomials [22].

Our strategy. Our approach draws a great deal of inspiration from Parusiński-Pragacz [34]. Indeed, all computations build on three fundamental computations of CSM classes, treated in [34]:

(a) the CSM class of the zero locus of a regular section of a vector bundle;
(b) the CSM class of a Grassmannian bundle; and
(c) the push-forward of the CSM class via a fibration.

In §2, we compute the *motivic refinement* of (a), (b), (c), see Lemmata 2.2, 2.3, 2.4. The CSM classes in cases (a), (b), (c) are recovered by specializing to $y = -1$. In order to make the generalization to Grassmannian and vexillary degeneracy loci possible, we combine (a), (b), (c) with the proof strategy introduced in [8] (see §3). Additional care is needed in dealing with motivic Chern and Hirzebruch classes when one generalizes methods designed for fundamental classes. Indeed, while the fundamental class coincides with the push-forward of the fundamental class of a resolution, motivic Chern and Hirzebruch classes are more delicate to handle. To arrive at the motivic Chern class of a vexillary degeneracy locus $W_{\boldsymbol{\tau}}$, we first compute the push-forward of the motivic Chern class of a resolution $\widetilde{\Omega}_{\boldsymbol{\tau}}$ of $W_{\boldsymbol{\tau}}$ (Theorem 3.1). Furthermore, using (c), we relate the push-forward of the motivic Chern class of $\widetilde{\Omega}_{\boldsymbol{\tau}}$ with $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ to the motivic Chern class of varieties $W_{\boldsymbol{\tau}^+}$ with $\boldsymbol{\tau}^+ = \left(\boldsymbol{k}^+, \boldsymbol{p}', \boldsymbol{q}'\right)$, for $\boldsymbol{k}^+ \geq \boldsymbol{k}'$ (Theorem 5.1). Finally, the motivic Chern class of $W_{\boldsymbol{\tau}}$ follows by the inclusion-exclusion principle.

A key step in our argument is the careful analysis in §5.1 of the stratification of a degeneracy locus induced from its resolution. We stratify $W_{\boldsymbol{\tau}}$ with $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ by the loci $W^{\circ}_{\boldsymbol{\tau}^+}$ with $\boldsymbol{\tau}^+ = \left(\boldsymbol{k}^+, \boldsymbol{p}', \boldsymbol{q}'\right)$, for $\boldsymbol{k}^+ \geq \boldsymbol{k}'$. These are precisely the strata on which the resolution $\widetilde{\Omega}_{\boldsymbol{\tau}} \to W_{\boldsymbol{\tau}}$ is locally trivial.

Theorem 5.1 uses the *additivity* of the motivic Hirzebruch class T_y as a transformation from the Grothendieck group $K_0(\mathrm{var}/X)$ of algebraic varieties over X. In fact, the Hirzebruch χ_y-genus is *the most general additive genus* [10]. Computation of other invariants, as the elliptic class and elliptic genus, would thus require new strategies.

Specializing to the case when the ambient variety X is a Grassmannian, our results give the motivic class of its Schubert varieties in terms of the motivic class of the Grassmannian. For instance, when $y = -1$, using the formula of [1] for the CSM class of Grassmannians, one can verify that the resulting formulas for the CSM class of Schubert varieties are consistent with the results in [1] after some nontrivial combinatorics (see §5.2).

Motivic Chern classes of Schubert cells in partial flag varieties have been computed in [18] via localization; the classes of the Schubert varieties could then be obtained from [18] by summing over all the strata of the closure of the Schubert cells. In the case of

vexillary permutations, our strategy produces the classes of Schubert varieties as a first outcome; this makes it feasible to arrive at a viable formula in the application to the Brill-Noether setting (see §7.3).

Open questions. In [6] we study more generally two-pointed Brill-Noether varieties, and show that they have the structure of determinantal varieties obtained from maps of flag bundles with rank conditions imposed by 321-avoiding permutations. We compute in [6] their connective K-theory class and holomorphic Euler characteristic. It would be interesting to compute their motivic Chern class, extending this work to the two-pointed case.

The holomorphic Euler characteristic of two-pointed Brill-Noether varieties is expressed in [14] as the enumeration of certain standard set-valued tableaux. We have found tableau formulas expressing the Hirzebruch χ_y-genus of one-pointed Brill-Noether surfaces. We wonder whether there exist tableau formulas expressing the Hirzebruch χ_y-genus of one- or two-pointed Brill-Noether varieties in general.

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1. Vexillary degeneracy loci

Here we set the notation and collect the assumptions used throughout. We adopt the notation of triples $\boldsymbol{\tau}$ from [8, 5].

1.1. **Vexillary degeneracy loci.** Let X be an irreducible variety over an algebraically closed field. Given maps of vector bundles over X as in (1) and a (weakly) increasing sequence $\boldsymbol{k} := (k_1, \dots, k_t)$ of positive integers, the *vexillary degeneracy locus* corresponding to the triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ is defined as

$$W_{\boldsymbol{\tau}} := \{x \in X \,:\, \dim \ker \left(E_{p_i} \to F_{q_i}\right)|_x \geq k_i \text{ for all } i\}$$

with inclusion $\iota\colon W_{\boldsymbol{\tau}} \hookrightarrow X$. Such a locus is Cohen-Macaulay when it has the expected dimension and when X is Cohen-Macaulay. We will compute the motivic Hirzebruch class of $W_{\boldsymbol{\tau}}$ in $A_*(X)[y]$, where $A_*(X)$ is the Chow group of X and y is a formal variable. It will be convenient to consider arbitrary weakly increasing sequences $\boldsymbol{k}$ (see e.g., §5.1).

1.1.1. *The reduced triple $\overline{\boldsymbol{\tau}}$.* Some of the conditions defining $W_{\boldsymbol{\tau}}$ may be redundant, and $W_{\boldsymbol{\tau}}$ could be similarly described as the vexillary degeneracy locus corresponding to a triple consisting of shorter sequences. After [19], a triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ with $\boldsymbol{p} = (0 \leq p_1 \leq \cdots \leq p_t)$ and $\boldsymbol{q} = (q_1 \geq \cdots \geq q_t \geq 0)$ is called *essential* if

$$0 < k_1 < \cdots < k_t \quad \text{and} \quad q_1 - p_1 + k_1 > \cdots > q_t - p_t + k_t > 0.$$

Given $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$, we denote by $\overline{\boldsymbol{\tau}} = \left(\overline{\boldsymbol{k}}, \overline{\boldsymbol{p}}, \overline{\boldsymbol{q}}\right)$ the essential triple of *shortest subsequences* $\overline{\boldsymbol{k}}$, $\overline{\boldsymbol{p}}$, and $\overline{\boldsymbol{q}}$ of $\boldsymbol{k}$, $\boldsymbol{p}$, and $\boldsymbol{q}$ such that $W_{\boldsymbol{\tau}} = W_{\overline{\boldsymbol{\tau}}}$. Necessarily, $\overline{\boldsymbol{k}}$ is strictly increasing.

Example 1.1. The sequences $\boldsymbol{k} = (2, 2, 3, 4)$, $\boldsymbol{p} = (4, 5, 6, 7)$, and $\boldsymbol{q} = (8, 7, 6, 3)$ and their subsequences $\overline{\boldsymbol{k}} = (2, 3)$, $\overline{\boldsymbol{p}} = (4, 6)$, and $\overline{\boldsymbol{q}} = (8, 6)$ describe the same degeneracy locus.

1.1.2. *The inflated triple $\boldsymbol{\tau}'$.* Assume the flag $E_{p_1} \subseteq E_{p_2} \subseteq \cdots \subseteq E_{p_t}$ extends to a full flag of sub-bundles $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{p_t}$ defined on X, and similarly, the flag $F_{q_1} \twoheadrightarrow F_{q_2} \twoheadrightarrow \cdots \twoheadrightarrow F_{q_t}$ extends to a full flag of quotients $F_{q_1} \twoheadrightarrow \cdots \twoheadrightarrow F_2 \twoheadrightarrow F_1$ defined on X. This assumption is indeed not restrictive, see Remark 3.3. Contrary to the previous subsection, we describe here how to inflate the triple $\boldsymbol{\tau}$ to a triple consisting of *longer* sequences defining the same locus $W_{\boldsymbol{\tau}}$. This will be used to define the locus $\widetilde{\Omega}_{\boldsymbol{\tau}}$ in §3.1.

Assume the triple $\boldsymbol{\tau}$ is essential, that is, $\boldsymbol{\tau} = \overline{\boldsymbol{\tau}}$. The triple $\boldsymbol{\tau}$ can then be inflated to a triple $\boldsymbol{\tau}' = (\boldsymbol{k}', \boldsymbol{p}', \boldsymbol{q}')$ by inflating the sequences $\boldsymbol{k}, \boldsymbol{p}$, and $\boldsymbol{q}$ of length t to sequences $\boldsymbol{k}', \boldsymbol{p}'$, and $\boldsymbol{q}'$ of length k_t with $\boldsymbol{k}' = (1, 2, \ldots, k_t)$ as in [8, §1.4] (see also further details in [5, §1]). Namely, suppose that $k_i > k_{i-1} + 1$, for some i. Then necessarily $p_i > p_{i-1}$ or $q_i < q_{i-1}$ (otherwise the corresponding conditions defining $W_{\boldsymbol{\tau}}$ are redundant, and the triple $\boldsymbol{\tau}$ can be reduced). When $p_i > p_{i-1}$, inflate by inserting the entry $k_i - 1$ between k_{i-1} and k_i in $\boldsymbol{k}$, the entry $p_i - 1$ between p_{i-1} and p_i in $\boldsymbol{p}$, and the entry q_i between q_{i-1} and q_i in $\boldsymbol{q}$. On the geometric side, the condition $\dim \ker (E_{p_i} \to F_{q_i})\,|_x \geq k_i$ implies $\dim \ker (E_{p_i - 1} \to F_{q_i})\,|_x \geq k_i - 1$ for a point $x \in W_{\boldsymbol{\tau}}$. The case $p_i = p_{i-1}$ and $q_i < q_{i-1}$ is treated similarly. Proceeding in this way, one arrives at sequences $\boldsymbol{k}', \boldsymbol{p}'$, and $\boldsymbol{q}'$ of length k_t such that $\boldsymbol{k}' = (1, 2, \ldots, k_t)$ and $W_{\boldsymbol{\tau}} = W_{\boldsymbol{\tau}'}$.

1.1.3. *Feasibility.* For an essential triple $\boldsymbol{\tau}$, in order for the conditions defining the locus $W_{\boldsymbol{\tau}}$ to be feasible, we assume that the sequence

$$\lambda_{k_i} := q_i - p_i + k_i \qquad \text{for } i = 1, \ldots, t, \tag{2}$$

is weakly decreasing.

1.2. **The partition $\boldsymbol{\lambda}_{\boldsymbol{\tau}}$.** Fix an essential triple $\boldsymbol{\tau}$. Extending (2), define the partition $\boldsymbol{\lambda}_{\boldsymbol{\tau}} = (\lambda_1, \ldots, \lambda_{k_t})$ as

$$\lambda_i := \lambda_{k_a} \quad \text{for } k_{a-1} < i \leq k_a.$$

For a triple $\boldsymbol{\tau}$ which is not necessarily essential, define $\boldsymbol{\lambda}_{\boldsymbol{\tau}} := \boldsymbol{\lambda}_{\overline{\boldsymbol{\tau}}}$, where $\overline{\boldsymbol{\tau}}$ is the reduced triple consisting of the shortest subsequences of $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$ such that $W_{\boldsymbol{\tau}} = W_{\overline{\boldsymbol{\tau}}}$ (§1.1.1). The expected codimension of the locus $W_{\boldsymbol{\tau}}$ in X is

$$\operatorname{codim}_X (W_{\boldsymbol{\tau}}) = |\boldsymbol{\lambda}_{\boldsymbol{\tau}}| := \sum_{i=1}^{k_t} \lambda_i.$$

Example 1.2. For the triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ where $\boldsymbol{k} = (2, 2, 3)$, $\boldsymbol{p} = (4, 5, 6)$, and $\boldsymbol{q} = (8, 7, 6)$, one has $\boldsymbol{\lambda}_{\boldsymbol{\tau}} = (6, 6, 3)$.

1.3. **The bundles $E(i)$, $F(i)$, classes $c(i)$, and operators $T_y(i)$.** For an essential triple $\boldsymbol{\tau}$, define the vector bundles $E(i)$ and $F(i)$ as

$$E(i) := E_{p_a} \quad \text{and} \quad F(i) := F_{q_a} \qquad \text{for } k_{a-1} < i \leq k_a.$$

For a triple $\boldsymbol{\tau}$ which is not necessarily essential, define

$$E(i) := E_{\overline{p}_a} \quad \text{and} \quad F(i) := F_{\overline{q}_a} \quad \text{for } \overline{k}_{a-1} < i \leq \overline{k}_a.$$

Here $\overline{\boldsymbol{k}}, \overline{\boldsymbol{p}}, \overline{\boldsymbol{q}}$ are the shortest subsequences of $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$ such that $W_{\boldsymbol{\tau}} = W_{\overline{\boldsymbol{\tau}}}$ with $\overline{\boldsymbol{\tau}} = \left(\overline{\boldsymbol{k}}, \overline{\boldsymbol{p}}, \overline{\boldsymbol{q}}\right)$ (see §1.1.1). Let

$$c(i) := c\left(F(i) - E(i)\right) = \frac{c\left(F(i)\right)}{c\left(E(i)\right)} \in A^*(X).$$

1.3.1. For each i, the *raising operator* R_i increases the index of the class $c(i)$ by one, that is,

$$R_i\, c(i)_m = c(i)_{m+1} \qquad \text{and} \qquad R_i\, c(k)_m = c(k)_m, \quad \text{for } k \neq i.$$

Moreover, R_i is extended linearly over $\mathbb{Q}$ and multiplicatively on monomials in classes $c(j)$, e.g., $R_i\left(c(j)_m\, c(k)_n\right) = \left(R_i\, c(j)_m\right)\left(R_i\, c(k)_n\right)$.

1.3.2. For a formal variable R and a vector bundle E of rank e with Chern roots a_i, for $i = 1, \ldots, e$, define

$$T_y\left(R \otimes E\right) := \prod_{i=1}^{e} Q_y\left(R + a_i\right), \text{ with } Q_y\left(\alpha\right) := \frac{\alpha(1+y)}{1 - e^{-\alpha(1+y)}} - \alpha y \in \mathbb{Q}[y][\![\alpha]\!].$$

As in (6), the terms in degree at most two are given by

$$\begin{aligned} T_y\left(R \otimes E\right) \;=\; & 1 + \frac{1}{2}(1-y)\left(eR + c_1(E)\right) \\ & + \frac{1}{12}(1+y)^2\left(eR^2 + 2c_1(E)R + \mathrm{ch}_2(E)\right) \\ & + \frac{1}{4}(1-y)^2\left(\binom{e}{2}R^2 + (e-1)c_1(E)R + c_2(E)\right) + \ldots \end{aligned}$$

where $\mathrm{ch}(E) = \sum_{i=1}^{e} \mathrm{ch}_i(E)$ is the Chern character of E. The motivic Hirzebruch class $T_y(X)$ of a smooth variety X is recovered when $R = 0$ and E is the tangent bundle of X (see §2). Similarly, in the absence of E, we set

$$T_y(R) := Q_y(R).$$

For each i, define the operator

$$T_y(i) := T_y\left(R_i \otimes \left(F(i) - E(i)\right)\right)$$

acting on the classes $c(j)$ such that R_i acts as in §1.3.1, and the Chern roots of the virtual bundle $F(i) - E(i)$ act by multiplication.

1.4. **Assumptions.** We collect here the assumptions used throughout. For a triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$, consider the vexillary degeneracy locus $W_{\boldsymbol{\tau}}$ and a sub-locus $W_{\boldsymbol{\tau}+} \subseteq W_{\boldsymbol{\tau}}$ where $\boldsymbol{\tau}^+ = \left(\boldsymbol{k}^+, \boldsymbol{p}', \boldsymbol{q}'\right)$ with $\boldsymbol{k}^+ \geq \boldsymbol{k}'$ in componentwise order. The corresponding locally closed stratum is

$$W^{\circ}_{\boldsymbol{\tau}+} := \left\{ x \in X \,:\, \dim \ker \left(E_{p'_i} \to F_{q'_i} \right) \big|_x = k_i^+ \right\}. \tag{3}$$

This is

$$W^{\circ}_{\boldsymbol{\tau}+} = W_{\boldsymbol{\tau}+} \smallsetminus \bigcup_{\boldsymbol{k}^{+\!\!+} > \boldsymbol{k}^+} W_{\boldsymbol{\tau}^{+\!\!+}} \subseteq X, \qquad \text{where } \boldsymbol{\tau}^{+\!\!+} = \left(\boldsymbol{k}^{+\!\!+}, \boldsymbol{p}', \boldsymbol{q}'\right).$$

We assume that X is an irreducible smooth algebraic variety over an algebraically closed field of characteristic zero, and for all weakly increasing sequences $\boldsymbol{k}^+ \geq \boldsymbol{k}'$ (i.e., $k_i^+ \geq i$

for all $i = 1, \dots, k_t$), the stratum $W^{\circ}_{\boldsymbol{\tau}^+}$ with $\boldsymbol{\tau}^+ = \left(\boldsymbol{k}^+, \boldsymbol{p}', \boldsymbol{q}'\right)$ is smooth of pure dimension $\dim X - |\boldsymbol{\lambda}_{\boldsymbol{\tau}^+}|$.

Remark 1.3. In [34], Parusiński-Pragacz show that their formula for the CSM class of the degeneracy locus of a single map between vector bundles holds under a weaker assumption which allows one to consider a *possibly singular* analytic variety X as ambient variety. It seems reasonable to expect that our results hold under a similar weaker assumption. For this, one needs to upgrade Lemma 2.2 below, for instance by working in the analytic category with a Whitney stratification of a singular X as in [34]. However, X will be smooth in all the applications we consider.

2. Motivic classes: fundamental computations

After briefly reviewing motivic Chern and Hirzebruch classes following [10], we discuss here three fundamental computations: Lemmata 2.2, 2.3, and 2.4. These will serve as the cornerstone of the paper.

For an algebraic variety X over a field of characteristic zero, let $K_0(\mathrm{var}/X)$ be the Grothendieck group of algebraic varieties over X, let $K_*(X)$ be the Grothendieck group of coherent sheaves of $\mathscr{O}_X$-modules, and $A_*(X)$ the Chow group. The transformations

$$\begin{array}{ccc} & K_0(\mathrm{var}/X) & \\ {\scriptstyle mC}\swarrow & & \searrow{\scriptstyle T_y} \\ K_*(X)\otimes\mathbb{Z}[y] & & A_*(X)\otimes\mathbb{Q}[y] \end{array}$$

are the unique transformations which commute with proper push-down and, for X smooth, satisfy

$$mC\left(\mathrm{id}_X\right) = \sum_{i\geq 0}\left[\wedge^i \mathscr{T}_X^{\vee}\right] y^i =: \lambda_y\left(\mathscr{T}_X^{\vee}\right), \tag{4}$$

$$T_y\left(\mathrm{id}_X\right) = \prod_{i=1}^{\dim X} Q_y(\alpha_i)\cap[X] =: T_y\left(\mathscr{T}_X\right)\cap[X]. \tag{5}$$

Here, α_i are the Chern roots of the tangent bundle $\mathscr{T}_X$, and $Q_y\left(\alpha\right)$ is the series

$$Q_y\left(\alpha\right) := \frac{\alpha(1+y)}{1-e^{-\alpha(1+y)}} - \alpha y \qquad \in \mathbb{Q}[y][\![\alpha]\!]$$

starting as

$$Q_y\left(\alpha\right) = 1 + \frac{1}{2}\alpha(1-y) + \frac{1}{12}\alpha^2(1+y)^2 + \dots. \tag{6}$$

The function λ_y satisfies: $\lambda_y(a+b) = \lambda_y(a)\lambda_y(b)$. The function T_y is the *motivic Hirzebruch class* function introduced in [25]. One has $T_y(a+b) = T_y(a)T_y(b)$, as well.

For arbitrary X, let $\{X_i\}_{i\in I}$ be a stratification of X, with X_i locally closed and smooth. By definition, we have

$$mC\left(\mathrm{id}_X\right) := \sum_{i\in I} mC\left(X_i \to X\right), \qquad T_y\left(\mathrm{id}_X\right) := \sum_{i\in I} T_y\left(X_i \to X\right). \tag{7}$$

Since any two such stratifications admit a common refinement, the above is well defined.

Remark 2.1. The transformations mC and T_y satisfy $td_{(1+y)} \circ mC = T_y$, where

$$td_{(1+y)} \colon K_*(X) \otimes \mathbb{Z}[y] \to A_*(X) \otimes \mathbb{Q}\left[y, (1+y)^{-1}\right]$$

is Yokura's generalization [36] of the Todd class transformation from the singular Riemann-Roch theorem [20].

2.1. **Zeros of sections.** For a vector bundle E on X, let $\iota \colon Z \hookrightarrow X$ be the zero locus of a regular section s of $E \to X$.

Lemma 2.2. *If X is smooth and $s \colon X \to E$ meets transversally the zero section of E, then*

$$mC\left(Z \hookrightarrow X\right) = \frac{\lambda_{-1}\left(E^\vee\right)}{\lambda_y\left(E^\vee\right)} mC\left(\mathrm{id}_X\right),$$
$$T_y\left(Z \hookrightarrow X\right) = \frac{c_{\mathrm{top}}(E)}{T_y(E)} T_y\left(\mathrm{id}_X\right).$$

Proof. Since s meets transversally the zero section of E, it follows that Z is smooth. We claim that

$$mC\left(Z \hookrightarrow X\right) = \frac{\lambda_{-1}\left(E|_X^\vee\right)}{\lambda_y\left(E|_X^\vee\right)} mC\left(\mathrm{id}_X\right).$$

Indeed, we have

$$mC\left(Z \hookrightarrow X\right) = \iota_! \, mC\left(\mathrm{id}_Z\right) = \iota_! \, \lambda_y\left(\mathscr{T}_Z^\vee\right) = \iota_! \left(\frac{\lambda_y\left(\mathscr{T}_Z^\vee\right)}{\iota^* \lambda_y\left(\mathscr{T}_X^\vee\right)}\right) \lambda_y\left(\mathscr{T}_X^\vee\right)$$
$$= \iota_! \left(\frac{1}{\iota^* \lambda_y\left(E|_X^\vee\right)}\right) \lambda_y\left(\mathscr{T}_X^\vee\right) = \frac{\lambda_{-1}\left(E|_X^\vee\right)}{\lambda_y\left(E|_X^\vee\right)} \lambda_y\left(\mathscr{T}_X^\vee\right).$$

Here, $\iota_!$ is the K-theoretic push-forward via ι. We have used the projection formula, and $\iota_!(1) = \lambda_{-1}(E|_X^\vee)$ in $K_*(X)$ (see for instance [21, V, Prop. 4.3]; this also appears in [18, §8.1]). The statement for mC follows. The same argument together with $\iota_*(1) = c_{\mathrm{top}}(E) \cap [X]$ in $A_*(X)$ prove the statement for T_y. □

When $y = -1$, Lemma 2.2 recovers the computation $\iota_* \, c_{\mathrm{SM}}(Z) = \frac{c_{\mathrm{top}}(E)}{c(E)} \cap c_{\mathrm{SM}}(X)$ treated in [34, Proposition 1.3].

2.2. **Grassmannian bundles.** Given a vector bundle E on X, let

$$\pi \colon \mathrm{Gr}(r, E) \to X$$

be the Grassmannian bundle parametrizing rank r sub-bundles of E. Consider the tautological exact sequence over $\mathrm{Gr}(r, E)$

$$0 \to S \to E \to Q \to 0.$$

Lemma 2.3. *For arbitrary X, we have*

$$mC\left(\mathrm{id}_{\mathrm{Gr}(r,E)}\right) = \lambda_y\left(\left(S^\vee \otimes Q\right)^\vee\right) \cdot \pi^* mC\left(\mathrm{id}_X\right),$$
$$T_y\left(\mathrm{id}_{\mathrm{Gr}(r,E)}\right) = T_y\left(S^\vee \otimes Q\right) \cdot \pi^* T_y\left(\mathrm{id}_X\right).$$

Proof. When X is smooth, $\mathrm{Gr}(r,E)$ is smooth. From the definition (4), we have

$$mC\left(\mathrm{id}_{\mathrm{Gr}(r,E)}\right) = \lambda_y\left(\mathscr{T}^\vee_{\mathrm{Gr}(r,E)}\right) = \lambda_y\left(\left(S^\vee \otimes Q\right)^\vee\right) \cdot \pi^* \lambda_y\left(\mathscr{T}^\vee_X\right),$$

and the statement about the motivic Chern class follows. For arbitrary X, the statement about the motivic Chern class is an immediate application of the Verdier-Riemann-Roch formula from [10, Cor. 2.1 (4)]. The proof for T_y is similar using [10, Cor. 3.1 (3)]. □

For $y = -1$, Lemma 2.3 recovers $c_{\mathrm{SM}}\left(\mathrm{Gr}(r,E)\right) = c\left(S^\vee \otimes Q\right) \cap \pi^* c_{\mathrm{SM}}(X)$, treated in [34, Proposition 1.5].

2.3. **Fibrations.** Given a proper morphism $p\colon Y \to X$, let $\mathscr{X} := \{X_k\}_{k\in K}$ be a stratification of X into locally closed strata X_k such that p is locally trivial in the Zariski topology over each X_k with smooth fiber F_k. Assume that there exists a unique top-dimensional stratum X_0 in $\mathscr{X}$. Let $p_!$ be the K-theoretic push-forward via p.

Lemma 2.4. *We have*

$$p_!\, mC\left(Y \to X\right) = \sum_{k\in K} d_k\, mC\left(\overline{X}_k \hookrightarrow X\right),$$
$$p_*\, T_y\left(Y \to X\right) = \sum_{k\in K} e_k\, T_y\left(\overline{X}_k \hookrightarrow X\right)$$

with

$$d_k := \left(\int_{F_k} \lambda_y\left(\mathscr{T}^\vee_{F_k}\right)\right) - \sum_j d_j, \qquad e_k := \left(\int_{F_k} T_y\left(F_k\right)\right) - \sum_j e_j$$

where the sums are over j such that $X_k \subset \overline{X}_j$.

Proof. Since p is locally trivial in the Zariski topology over each X_k with smooth fiber F_k, factoring $p^{-1}(X_k) \subset Y \xrightarrow{p} X$ as $p^{-1}(X_k) \xrightarrow{p} X_k \hookrightarrow X$ and by multiplicativity of λ_y, one has

$$p_!\, mC\left(p^{-1}(X_k) \to X\right) = \left(\int_{F_k} \lambda_y\left(\mathscr{T}^\vee_{F_k}\right)\right) mC\left(X_k \hookrightarrow X\right),$$

for each $k \in K$. Therefore, one has $d_0 = \int_{F_0} \lambda_y\left(\mathscr{T}^\vee_{F_0}\right)$. One verifies the formula for the coefficient d_k by recursion on the codimension of the strata in $\mathscr{X}$. The proof for T_y is identical. □

In the case $y = -1$, Lemma 2.3 was treated in [34, Proposition 1.6].

2.4. **Hirzebruch χ_y-genus.** For a compact X, its *Hirzebruch χ_y-genus* is

$$\chi_y(X) := \int_X T_y(X) \qquad \in \mathbb{Q}[y].$$

This invariant recovers the topological Euler characteristic for $y = -1$, the holomorphic Euler characteristic for $y = 0$, and the signature for $y = 1$ [10]. The Hirzebruch χ_y-genus extends to arbitrary X via the additivity in $K_0(\mathrm{var/pt})$.

Example 2.5. For the projective space $\mathbb{CP}^n$, one has

$$\chi_y\left(\mathbb{CP}^n\right) = \int_{\mathbb{CP}^n} T_y\left(\mathbb{CP}^n\right) = 1 + (-y) + \cdots + (-y)^n.$$

Consequently, one has $\chi_y\left(\mathbb{A}^n_{\mathbb{C}}\right) = (-y)^n$.

3. Motivic classes of a resolution of vexillary degeneracy loci

Since motivic Chern and Hirzebruch classes have an equivalent formalism, as exemplified by Lemmata 2.2, 2.3, 2.4, for simplicity we consider only the case of motivic Hirzebruch classes from this point on. The main result of this section is the following Theorem 3.1, computing the motivic Hirzebruch class of a *resolution* of vexillary degeneracy loci. This is the explicit version of Theorem 1.

3.1. **The locus $\widetilde{\Omega}_{\boldsymbol{\tau}}$.** Recall the geometric setup of §1.1 defining the vexillary degeneracy locus $W_{\boldsymbol{\tau}}$ in a variety X given maps of vector bundles (1) and a weakly increasing sequence $\boldsymbol{k} = (k_1, \ldots, k_t)$. We define here a resolution $\widetilde{\Omega}_{\boldsymbol{\tau}}$ of $W_{\boldsymbol{\tau}}$.

As in §1.1.1, we can reduce to the case when the triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ is essential, that is, it corresponds to a minimal set of conditions $\dim \ker \left(E_{p_i} \to F_{q_i}\right)|_x \geq k_i$ for a point $x \in W_{\boldsymbol{\tau}}$. The triple $\boldsymbol{\tau}$ could then be inflated to the triple $\boldsymbol{\tau}'$ as in §1.1.2 and $W_{\boldsymbol{\tau}} \cong W_{\boldsymbol{\tau}'}$.

Consider the variety X_{k_t} parametrizing full flags of sub-bundles $V_1 \subseteq \cdots \subseteq V_{k_t}$ with $\mathrm{rank}(V_i) = i$ and $V_i \subseteq E_{p'_i}$ for each i. The variety X_{k_t} is constructed as a sequence of projective bundles

$$X =: X_0 \xleftarrow{\pi_1} \mathbb{P}\left(E_{p'_1}\right) =: X_1 \xleftarrow{\pi_2} \mathbb{P}\left(E_{p'_2}/\mathbb{S}_1\right) =: X_2 \cdots \xleftarrow{\pi_{k_t}} \mathbb{P}\left(E_{p'_{k_t}}/\mathbb{S}_{k_t-1}\right) =: X_{k_t},$$

where $\mathbb{S}_i/\mathbb{S}_{i-1}$ is the tautological line bundle on X_i, for each i. Here, we omit the obvious pull-backs via the natural projections π_i to simplify the notation. Since X is assumed to be smooth, X_i is also smooth, for each i. Let $\pi\colon X_{k_t} \to X$ be the natural projection. Define

$$\widetilde{\Omega}_{\boldsymbol{\tau}} := \left\{(x, V_1 \subseteq \cdots \subseteq V_{k_t}) \in X_{k_t} \,:\, V_i \subseteq \ker\left(E_{p'_i} \to F_{q'_i}\right)\Big|_x \text{ for all } i\right\}$$

with natural inclusion $\iota\colon \widetilde{\Omega}_{\boldsymbol{\tau}} \hookrightarrow X_{k_t}$. (We study a quotient $\widetilde{\Omega} \to \Omega$ in §6, hence the use of the tilde here.) The restriction ϕ of π to $\widetilde{\Omega}_{\boldsymbol{\tau}}$ is a resolution of singularities as in Kempf-Laksov [30]. One has a commutative diagram

$$\begin{array}{ccc} \widetilde{\Omega}_{\boldsymbol{\tau}} & \xhookrightarrow{\iota} & X_{k_t} \\ \phi\downarrow & & \downarrow\pi \\ W_{\boldsymbol{\tau}} & \hookrightarrow & X \end{array}$$

and the fiber of ϕ over a point x in $W_{\boldsymbol{\tau}}$ is

$$\left\{(V_1 \subseteq \cdots \subseteq V_{k_t}) \in \pi^{-1}(x) \,:\, V_i \subseteq \ker\left(E_{p'_i} \to F_{q'_i}\right)\Big|_x \text{ for all } i\right\}.$$

Note that since $(\boldsymbol{\tau}')' = \boldsymbol{\tau}'$, one has $\widetilde{\Omega}_{\boldsymbol{\tau}} = \widetilde{\Omega}_{\boldsymbol{\tau}'}$ by definition.

In general, the locus $\widetilde{\Omega}_{\boldsymbol{\tau}}$ is not the *minimal* resolution of $W_{\boldsymbol{\tau}}$. The minimal resolution of $W_{\boldsymbol{\tau}}$ is given by the variety parametrizing flags $V_{k_1} \subseteq \cdots \subseteq V_{k_t}$ such that $\operatorname{rank}(V_{k_i}) = k_i$ and $V_{k_i} \subseteq \ker(E_{p_i} \to F_{q_i})$ for each i. There is a forgetful map from $\widetilde{\Omega}_{\boldsymbol{\tau}}$ to the minimal resolution of $W_{\boldsymbol{\tau}}$ which is birational. We find a closed formula precisely for the (push-forward of the) motivic Hirzebruch class of $\widetilde{\Omega}_{\boldsymbol{\tau}}$.

Define $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\boldsymbol{\tau}}$ as in §1.2, and bundles $E(i), F(i)$, classes $c(i)$, and operators R_i, $T_y(R_i)$, and $T_y(i)$ as in §1.3. Let $c(i)_j$ be the term of degree j in $c(i)$.

Theorem 3.1. *With assumptions as in §1.4, the class* $(\pi\iota)_* T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right)$ *is*

$$\frac{1}{\prod_{(i,j)\in S} T_y(R_j - R_i)} \left| \frac{1}{T_y(i)} c(i)_{\lambda_i + j - i} \right|_{1 \le i,j \le k_t} \cap T_y(X),$$

where $S := \{(i,j) : i \le k_a < j, \text{ for some } a\}$ *and* $T_y(R_j - R_i) := Q_y(R_j - R_i)$. *Equivalently, this is*

$$\frac{1}{\prod_{(i,j)\in S} T_y(R_j - R_i)} \prod_{i=1}^{k_t} \frac{1}{T_y(i)} \iota_* [W_{\boldsymbol{\tau}}] \cap T_y(X).$$

The second expression in the statement follows from the first since

$$(\pi\iota)_* \left[\widetilde{\Omega}_{\boldsymbol{\tau}}\right] = \iota_* [W_{\boldsymbol{\tau}}] = \left| c_{\lambda_i + j - i}(i) \right|_{1 \le i,j \le k_t} \in A^*(X)$$

by [19]. For $y = -1$, the class $(\pi\iota)_* c_{\mathrm{SM}}\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right)$ admits a simpler expression, see §4.

Remark 3.2. We emphasize that all raising operators in Theorem 3.1 apply to the Chern classes in the expansion of $[W_{\boldsymbol{\tau}}]$, and do not apply to the Chern classes contributed from the operators $T_y(i)$.

The first few terms in the expansion of the operator in Theorem 3.1 are given by multiplying

$$\frac{1}{T_y(R_j - R_i)} = 1 + \frac{1}{2}(y-1)(R_j - R_i) + \frac{1}{6}(y^2 - 4y + 1)(R_j - R_i)^2 + \dots$$

for $(i,j) \in S$, and

$$\begin{aligned} \frac{1}{T_y(i)} \;=\; & 1 + \frac{1}{2}(y-1)\left(\mathrm{ch}_0(i) R_i + c_1(i)\right) \\ & - \frac{1}{12}(1+y)^2\left(\mathrm{ch}_0(i) R_i^2 + 2c_1(i) R_i + \mathrm{ch}_2(i)\right) \\ & + \frac{1}{4}(1-y)^2\Bigg(\binom{\mathrm{ch}_0(i)+1}{2} R_i^2 + (\mathrm{ch}_0(i)+1) c_1(i) R_i \\ & \qquad + c_2(E(i) - F(i))\Bigg) + \dots \end{aligned}$$

where $\mathrm{ch}(i) := \mathrm{ch}(F(i) - E(i))$, for each i.

As an example for the set S, for $\boldsymbol{k} = (2, 5)$, one has

$$S = \{(1,3), (2,3), (1,4), (2,4), (1,5), (2,5)\}.$$

Remark 3.3. The definition of the locus $\widetilde{\Omega}_{\boldsymbol{\tau}}$ requires flags of vector bundles finer than the ones given in (1) precisely when $k_i > i$ for some i. However, the expression in the statement only depends on the two given flags in (1), and not on the choice of finer flags $E_{p'_1} \subseteq \cdots \subseteq E_{p'_{k_t}}$ and $F_{q'_1} \twoheadrightarrow \cdots \twoheadrightarrow F_{q'_{k_t}}$ used to define the locus $\widetilde{\Omega}_{\boldsymbol{\tau}}$.

In fact, the flags in (1) can be extended to full flags of vector bundles after passing to an appropriate projective bundle over X. Namely, suppose that $p_i > p_{i-1}$. To define a vector bundle E_{p_i-1} of rank $p_i - 1$ such that $E_{p_{i-1}} \subset E_{p_i-1} \subset E_{p_i}$, consider the projective bundle

$$\mathbb{P}\left(E_{p_i}/E_{p_{i-1}}\right) \to X \tag{8}$$

and set $E_{p_i-1} := Q$, where $Q/E_{p_{i-1}}$ is the universal quotient bundle on $\mathbb{P}\left(E_{p_i}/E_{p_{i-1}}\right)$ — here again we omit to denote when a bundle is pulled back for simplicity. Proceeding in this way, one constructs full flags $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{p_t}$ and $F_{q_1} \twoheadrightarrow \cdots \twoheadrightarrow F_2 \twoheadrightarrow F_1$ defined on a space X' obtained as a tower of projective bundles $p\colon X' \to X$.

The theorem then applies to give the class

$$\frac{1}{\prod_{(i,j)\in S} T_y\left(R_j - R_i\right)} \left| \frac{1}{T_y\left(i\right)} c\left(i\right)_{\lambda_i + j - i} \right|_{1 \le i,j \le k_t} \cap T_y\left(X'\right) \quad \in A_*\left(X'\right)[y]. \tag{9}$$

Factoring p as a composition of projective bundles, $A_*(X)$ can be identified as a subring of $A_*\left(X'\right)$ via p^*. After Lemma 2.2, the class $\left(\pi\iota\right)_* T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right)$ in $p^*\left(A_*(X)[y]\right)$ is recovered after quotienting (9) by the motivic Hirzebruch class of the consecutive fibers (the fibers of p are locally constant). Indeed, this has the effect of replacing $T_y\left(X'\right)$ with $p^* T_y\left(X\right)$ in (9), thus it recovers the formula in the theorem.

To prove the theorem, we distinguish four cases, following the proof strategy from [8]. In the basic and dominant cases (§§3.2–3.3), we immediately compute the motivic Hirzebruch class of the corresponding degeneracy loci. In the main and general cases (§§3.4–3.5), where the degeneracy loci are singular, we compute here the motivic Hirzebruch class of the *push-forward of the resolution* $\widetilde{\Omega}$ of the degeneracy loci. In the last two cases, the computation of the motivic Hirzebruch class of the actual degeneracy loci is completed as in Theorem 3 building on Theorem 3.1.

3.2. **Basic case.** Let L be a line bundle and E a vector bundle of rank e on X.

Lemma 3.4. *If the locus $\iota\colon Z \hookrightarrow X$ where a map $\varphi\colon L \to E$ vanishes is smooth of codimension e, its motivic Hirzebruch class is*

$$\iota_* T_y(Z) = \frac{1}{T_y\left(E \otimes R\right)} c_e\left(E - L\right) \cap T_y\left(X\right),$$

where R is the raising operator acting on $c\left(E - L\right)$.

Proof. From Lemma 2.2, we have

$$\iota_* T_y(Z) = \frac{c_e\left(E \otimes L^\vee\right)}{T_y\left(E \otimes L^\vee\right)} \cap T_y\left(X\right).$$

The statement follows from the identities $c_e\left(E \otimes L^\vee\right) = c_e\left(E - L\right)$ and

$$\left(c_1\left(L^\vee\right)\right)^k c_i\left(E - L\right) = c_{i+k}\left(E - L\right), \qquad \text{for } k \geq 0 \text{ and } i \geq e \tag{10}$$

(see e.g., [8, pg. 3]), so that multiplication by $c_1\left(L^\vee\right)$ here coincides with the operator R. □

3.3. **Dominant case.** Consider maps of vector bundles on X

$$E_1 \hookrightarrow \cdots \hookrightarrow E_t \to F_{q_1} \twoheadrightarrow \cdots \twoheadrightarrow F_{q_t}$$

with $\operatorname{rank}(E_i) = i$ and $\operatorname{rank}(F_{q_i}) = q_i$. Here, we assume that $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ is a triple with $k_i = p_i = i$, for $1 \leq i \leq t$. Consider the degeneracy locus

$$W_{\boldsymbol{\tau}} := \{x \in X \,:\, (E_i \to F_{q_i})\,|_x \text{ is zero for all } i\}$$

with inclusion $\iota\colon W_{\boldsymbol{\tau}} \hookrightarrow X$. In this case, $k_i = i$ and $\lambda_i = q_i$, for each i.

Lemma 3.5. *When $W_{\boldsymbol{\tau}}$ is smooth of dimension $\dim X - \boldsymbol{\lambda}_{\boldsymbol{\tau}}$, the class $\iota_* T_y(W_{\boldsymbol{\tau}})$ is given by*

$$\left|\frac{1}{T_y\left(R_i \otimes F_{q_i}\right)} c_{\lambda_i + j - i}\left(F_{q_i} - E_i\right)\right|_{1 \leq i,j \leq t} \cap T_y\left(X\right).$$

Proof. From Lemma 3.4, it follows that

$$\iota_* T_y\left(W_{\boldsymbol{\tau}}\right) = \prod_{i=1}^{t} \frac{1}{T_y\left(F_{q_i} \otimes \left(E_i/E_{i-1}\right)^\vee\right)} c_{\lambda_i}\left(F_{q_i} - E_i/E_{i-1}\right) \cap T_y\left(X\right).$$

As in (10), one has that $c_1((E_i/E_{i-1})^\vee)$ acts as R_i on $c_{\lambda_i}(F_{q_i} - E_i/E_{i-1})$. By means of the identity

$$\prod_{i=1}^{t} c_{\lambda_i}\left(F_{q_i} - E_i/E_{i-1}\right) = \left|c_{\lambda_i + j - i}\left(F_{q_i} - E_i\right)\right|_{1 \leq i,j \leq t}, \tag{11}$$

valid as in [8, §1.2], the statement follows. □

3.4. **Main case.** Consider maps of vector bundles on X

$$E_{p_1} \hookrightarrow \cdots \hookrightarrow E_{p_t} \to F_{q_1} \twoheadrightarrow \cdots \twoheadrightarrow F_{q_t}$$

with $\operatorname{rank}(E_{p_i}) = p_i$ and $\operatorname{rank}(F_{q_i}) = q_i$. Here, we assume that $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ is a triple where $k_i = i$, for $1 \leq i \leq t$, and $q_1 - p_1 + 1 > \cdots > q_t - p_t + t$. In particular, the triple $\boldsymbol{\tau}$ is essential and $\boldsymbol{\tau} = \boldsymbol{\tau}'$ (notation as in §1.1). We would like to compute the motivic Hirzebruch class of the locus

$$W_{\boldsymbol{\tau}} := \{x \in X \,:\, \dim \ker\left(E_{p_i} \to F_{q_i}\right)|_x \geq i\}.$$

Instead, we compute here the motivic Hirzebruch class of the *resolution* of $W_{\boldsymbol{\tau}}$; this will be used in §5 to obtain the motivic Hirzebruch class of $W_{\boldsymbol{\tau}}$.

Consider the following sequence of projective bundles

$$X =: X_0 \xleftarrow{\pi_1} \mathbb{P}\left(E_{p_1}\right) =: X_1 \xleftarrow{\pi_2} \mathbb{P}\left(E_{p_2}/\mathbb{S}_1\right) =: X_2 \cdots \xleftarrow{\pi_t} \mathbb{P}\left(E_{p_t}/\mathbb{S}_{t-1}\right) =: X_t,$$

where $\mathbb{S}_i/\mathbb{S}_{i-1}$ is the tautological line bundle on X_i. The variety X_t here parametrizes flags of sub-bundles $V_1 \subset \cdots \subset V_t$ with $\operatorname{rank}(V_i) = i$ and $V_i \subseteq E_{p_i}$. Since X is assumed to be smooth, X_i is also smooth, for each i. Define

$$\widetilde{\Omega}_{\boldsymbol{\tau}} := \{(x, V_1 \subset \cdots \subset V_t) \in X_t \,:\, V_i \subseteq \ker\left(E_{p_i} \to F_{q_i}\right)|_x \text{ for all } i\}$$

with natural maps $\iota\colon \widetilde{\Omega}_{\boldsymbol{\tau}} \hookrightarrow X_t$ and $\pi\colon X_t \to X$.

Proposition 3.6. *With assumptions as in §1.4, the class* $(\pi\iota)_* T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right)$ *is*

$$\frac{1}{\prod_{i<j} T_y\left(R_j - R_i\right)} \left| \frac{1}{T_y\left(R_i \otimes \left(F_{q_i} - E_{p_i}\right)\right)} c_{\lambda_i+j-i}\left(F_{q_i} - E_{p_i}\right) \right|_{1\le i,j\le t} \cap T_y(X).$$

Equivalently, this is

$$\frac{1}{\prod_{i<j} T_y\left(R_j - R_i\right)} \prod_{i=1}^{t} \frac{1}{T_y\left(R_i \otimes \left(F_{q_i} - E_{p_i}\right)\right)} \left[W_{\boldsymbol{\tau}}\right] \cap T_y(X).$$

Note that for $\boldsymbol{k} = (1, \ldots, t)$, one has $E(i) = E_{p_i}$, $F(i) = F_{q_i}$, and $S = \{(i,j) : i < j\}$, hence the formula in Theorem 3.1 specializes to the one in Proposition 3.6.

Proof. From Lemma 2.3, we have

$$T_y(X_i) = T_y\left(\left(\mathbb{S}_i/\mathbb{S}_{i-1}\right)^\vee \otimes E_{p_i}/\mathbb{S}_i\right) \cap \pi_i^* T_y\left(X_{i-1}\right).$$

Combining this with the basic case, one has

$$\iota_* T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right) = \prod_{i=1}^{t} \frac{T_y\left(\left(\mathbb{S}_i/\mathbb{S}_{i-1}\right)^\vee \otimes E_{p_i}/\mathbb{S}_i\right)}{T_y\left(\left(\mathbb{S}_i/\mathbb{S}_{i-1}\right)^\vee \otimes F_{q_i}\right)} c_{q_i}\left(F_{q_i} - \mathbb{S}_i/\mathbb{S}_{i-1}\right) \cap \pi^* T_y(X).$$

As in (10), one has that $c_1((\mathbb{S}_i/\mathbb{S}_{i-1})^\vee)$ acts as R_i on $c_{q_i}(F_{q_i} - \mathbb{S}_i/\mathbb{S}_{i-1})$. The statement follows since one has

$$\begin{aligned} T_y\left(R_i \otimes \mathbb{S}_i\right) &= T_y\left(R_i - c_1\left(\mathbb{S}_1^\vee\right)\right) \cdots T_y\left(R_i - c_1\left(\left(\mathbb{S}_i/\mathbb{S}_{i-1}\right)^\vee\right)\right) \\ &= T_y\left(R_i - R_1\right) \cdots T_y\left(R_i - R_{i-1}\right), \end{aligned}$$

and

$$\prod_{i=1}^{t} c_{q_i}\left(F_{q_i} - \mathbb{S}_i/\mathbb{S}_{i-1}\right) = \left|c_{q_i+j-i}\left(F_{q_i} - \mathbb{S}_i\right)\right|_{1\le i,j\le t}$$

as in (11), and

$$\pi_* \left|c_{q_i+j-i}\left(F_{q_i} - \mathbb{S}_i\right)\right|_{1\le i,j\le t} = \left|c_{\lambda_i+j-i}\left(F_{q_i} - E_{p_i}\right)\right|_{1\le i,j\le t}$$

as in [8, §1.3]. □

3.5. **General case.** We consider here the general case of Theorem 3.1 and thus complete its proof.

Proof of Theorem 3.1. After §1.1.1, we can restrict to the case of essential triples $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$. Indeed, the formula in the statement remains invariant after replacing a triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ with $\overline{\boldsymbol{\tau}} = \left(\overline{\boldsymbol{k}}, \overline{\boldsymbol{p}}, \overline{\boldsymbol{q}}\right)$ such that $\overline{\boldsymbol{\tau}}$ is essential and $W_{\boldsymbol{\tau}} = W_{\overline{\boldsymbol{\tau}}}$, as in §1.1.1, and thus $\widetilde{\Omega}_{\boldsymbol{\tau}} \cong \widetilde{\Omega}_{\overline{\boldsymbol{\tau}}}$.

The case when $\boldsymbol{k} = (1, 2, \ldots, t)$ is treated by Proposition 3.6. Otherwise, to define $\widetilde{\Omega}_{\boldsymbol{\tau}}$, the triple $\boldsymbol{\tau}$ is inflated to a triple $\boldsymbol{\tau}' = (\boldsymbol{k}', \boldsymbol{p}', \boldsymbol{q}')$ such that the sequences $\boldsymbol{k}', \boldsymbol{p}'$, and $\boldsymbol{q}'$ are of length k_t with $\boldsymbol{k}' = (1, 2, \ldots, k_t)$. One has $\boldsymbol{\lambda}_{\boldsymbol{\tau}} = \boldsymbol{\lambda}_{\boldsymbol{\tau}'}$ and $\widetilde{\Omega}_{\boldsymbol{\tau}} = \widetilde{\Omega}_{\boldsymbol{\tau}'}$ by definition. We can thus apply Proposition 3.6 to compute $(\pi\iota)_* T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}'}\right)$ with respect to the flag $E_{p'_1} \subseteq \cdots \subseteq E_{p'_{k_t}}$ refining $E_{p_1} \subseteq \cdots \subseteq E_{p_t}$, and the flag $F_{q'_1} \twoheadrightarrow \cdots \twoheadrightarrow F_{q'_{k_t}}$ refining $F_{q_1} \twoheadrightarrow \cdots \twoheadrightarrow F_{q_t}$. After Remark 3.3, we can assume that such finer flags exist on X. Thus from Proposition 3.6, the class $(\pi\iota)_* T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}'}\right)$ is

$$\frac{1}{\prod_{i<j} T_y\left(R_j - R_i\right)} \left| \frac{1}{T_y\left(R_i \otimes \left(F_{q'_i} - E_{p'_i}\right)\right)} c_{\lambda_i + j - i}\left(F_{q'_i} - E_{p'_i}\right) \right|_{1 \leq i, j \leq k_t} \cap T_y(X).$$

It remains to verify that this formula is equivalent to the formula in the statement.

As in [8, §1.4] (see also [5, §1]), one has

$$|c(i)_{\lambda_i + j - i}|_{1 \leq i, j \leq k_t} = \left| c_{\lambda_i + j - i}(F_{q'_i} - E_{p'_i}) \right|_{1 \leq i, j \leq k_t}. \tag{12}$$

Indeed, from the definition of $c(i)$ (see §1.3), the entries of the two determinants in (12) do not match, but the determinants do. Furthermore, fix i with $1 \leq i \leq k_t$. One has $k_{a-1} < i \leq k_a$ for some a. We claim that

$$\prod_{j : j > k_a} \frac{1}{T_y\left(R_j - R_i\right)} \cdot \frac{1}{T_y(i)} \quad = \quad \prod_{j : j > i} \frac{1}{T_y\left(R_j - R_i\right)} \cdot \frac{1}{T_y\left(R_i \otimes \left(F_{q'_i} - E_{p'_i}\right)\right)} \tag{13}$$

as operators on (12). Recall that $T_y(i) := T_y\left(R_i \otimes (F(i) - E(i))\right)$, where $E(i) = E_{p_a}$ and $F(i) = F_{q_a}$, by definition (§1.3). Assume that $q'_i = q_a$. Then one has

$$q'_i = q'_{i+1} = \cdots = q'_{k_a} = q_a \quad \text{and} \quad \left(p'_i, p'_{i+1}, \ldots, p'_{k_a}\right) = (p_a - k_a + i, \ldots, p_a - 1, p_a)$$

by construction. It follows that

$$\frac{1}{T_y\left(i\right)} = \frac{1}{T_y\left(R_i \otimes \left(F_{q'_i} - E_{p'_{k_a}}\right)\right)} = \frac{T_y\left(R_i \otimes \left(E_{p'_{k_a}} / E_{p'_i}\right)\right)}{T_y\left(R_i \otimes \left(F_{q'_i} - E_{p'_i}\right)\right)}.$$

The second equality follows from the multiplicativity of T_y on exact sequences. Furthermore, one has

$$T_y\left(E_{p'_{k_a}} / E_{p'_i}\right) = \prod_{j : i < j \leq k_a} T_y\left(c_1\left(E_{p'_j} / E_{p'_{j-1}}\right)\right) = \prod_{j : i < j \leq k_a} T_y\left(-R_j\right)$$

as operators on (12). The second equality follows from (10): for j such that $i < j \leq k_a$, the indices of the Chern class $c\left(F_{q'_j} - E_{p'_j}\right) = c\left(F_{q'_j} - E_{p'_{j-1}} - E_{p'_j}/E_{p'_{j-1}}\right)$ in the j-th row of the right-hand side of (12) are at least

$$\lambda_j + 1 - j = q'_j - p'_j + 1 = q'_j - p'_{j-1} = \operatorname{rank}\left(F_{q'_j} - E_{p'_{j-1}}\right),$$

hence from (10), $c_1\left(E_{p'_j}/E_{p'_{j-1}}\right)$ acts as the operator $-R_j$ on (12). Then, one has

$$T_y\left(R_i \otimes \left(E_{p'_{k_a}}/E_{p'_i}\right)\right) = \prod_{j:i<j\leq k_a} T_y\left(R_i \otimes c_1\left(E_{p'_j}/E_{p'_{j-1}}\right)\right) = \prod_{j:i<j\leq k_a} T_y\left(R_i - R_j\right)$$

as operators on (12). It follows that

$$\frac{1}{T_y\left(i\right)} = \prod_{j:i<j\leq k_a} \frac{1}{T_y\left(R_j - R_i\right)} \cdot \frac{1}{T_y\left(R_i \otimes \left(F_{q'_i} - E_{p'_i}\right)\right)}$$

as operators on (12), whence the claim (13). When $q'_i < q_a$, then necessarily $p_a = p'_i$, and this case is treated similarly. We thus conclude that the formula for $(\pi\iota)_* T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}'}\right)$ from Proposition 3.6 is equivalent to the formula in the statement. □

4. CSM classes of a resolution of vexillary degeneracy loci

In the case of the CSM class, the results of the previous section simplify as follows. For a triple $\boldsymbol{\tau}$, define $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\boldsymbol{\tau}}$ as in §1.2, and bundles $E(i), F(i)$, classes $c(i)$, and operators R_i and $T_y(i)$ as in §1.3. Let $c(i)_j$ be the term of degree j in $c(i)$, and given a variable t, define $c_t(i) := \sum_{j\geq 0} c(i)_j\, t^j$. We use below the virtual rank $\operatorname{ch}(i)_0 = \operatorname{rank}(F(i) - E(i))$.

Theorem 4.1. *With assumptions as in §1.4, the class* $(\pi\iota)_* c_{\mathrm{SM}}\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right)$ *is given by*

$$\prod_{(i,j)\in S} \frac{1}{1 + R_j - R_i} \left| \frac{(1+R_i)^{-\operatorname{ch}(i)_0}}{c_{\frac{1}{1+R_i}}(i)} c(i)_{\lambda_i + j - i} \right|_{1\leq i,j\leq k_t} \cap c_{\mathrm{SM}}\left(X\right),$$

where $S := \{(i,j) : i \leq k_a < j \text{ for some } a\}$. *Equivalently, this is*

$$\prod_{(i,j)\in S} \frac{1}{1 + R_j - R_i} \prod_{i=1}^{t} \frac{(1+R_i)^{-\operatorname{ch}(i)_0}}{c_{\frac{1}{1+R_i}}(i)} \left[W_{\boldsymbol{\tau}}\right] \cap c_{\mathrm{SM}}\left(X\right).$$

We emphasize that in the above formula all raising operators apply to the Chern classes coming from the expressions of type $c(i)_{\lambda_i+j-i}$ in the expansion of $[W_{\boldsymbol{\tau}}]$, and do not apply to the Chern classes contributed from the terms $c_{\frac{1}{1+R_i}}(i)$.

The first few terms in the expansion of the operator in Theorem 4.1 are given by multiplying

$$\frac{1}{1 + R_j - R_i} = 1 - (R_j - R_i) + (R_j - R_i)^2 + \dots$$

for $(i,j) \in S$, and

$$\begin{aligned} \frac{1}{c_{\frac{1}{1+R_i}}(i)} &= 1 - \frac{c(i)_1}{1+R_i} + \left(c(i)_1^2 - c(i)_2\right)\left(\frac{1}{1+R_i}\right)^2 + \dots \\ &= 1 - c(i)_1 + c(i)_1 R_i + c(i)_1^2 - c(i)_2 + \cdots \end{aligned}$$

for each i, and

$$(1+R_i)^{-\mathrm{ch}(i)_0} = 1 - \mathrm{ch}(i)_0 R_i + \frac{-\mathrm{ch}(i)_0(-\mathrm{ch}(i)_0 - 1)}{2} R_i^2 + \cdots$$

for each i.

To show Theorem 4.1, we use the following technical lemma:

Lemma 4.2. *Let L be a line bundle and E a virtual vector bundle of virtual rank e. One has*

$$c(E \otimes L) = (1 + c_1(L))^e c_{\frac{1}{1+c_1(L)}}(E).$$

Here, $c_t(E) := \sum_{i \geq 0} c_i(E)\, t^i$. The case when E is a vector bundle is [20, Example 3.2.2]. For a virtual vector bundle, the argument is similar.

Proof of Theorem 4.1. Specializing Theorem 3.1 at $y = -1$, we have that the class $(\pi\iota)_* \, c_{\mathrm{SM}}\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right)$ is given by

$$\prod_{(i,j)\in S} \frac{1}{1 + R_j - R_i} \left| \frac{1}{c\left(R_i \otimes (F(i) - E(i))\right)} c(i)_{\lambda_i + j - i} \right|_{1 \leq i,j \leq k_t} \cap c_{\mathrm{SM}}(X),$$

where $c\left(R_i \otimes (F(i) - E(i))\right) := T_{-1}(i)$ is the specialization at $y = -1$ of the operator $T_y(i)$ from §1.3. The virtual bundle $F(i) - E(i)$ has virtual rank equal to $\mathrm{ch}(i)_0 = \mathrm{rank}\,(F(i)) - \mathrm{rank}\,(E(i))$. Applying Lemma 4.2, we have

$$c\left(R_i \otimes (F(i) - E(i))\right) = (1+R_i)^{\mathrm{ch}(i)_0} \, c_{\frac{1}{1+R_i}}(i),$$

hence the statement. □

5. Motivic classes of vexillary degeneracy loci

For a triple $\boldsymbol{\tau}$, recall the inflated triple $\boldsymbol{\tau}'$ from §1.1.2. The aim of this section is to prove Theorem 2, here restated:

Theorem 5.1. *For a triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ and with assumptions as in §1.4, one has*

$$(\pi\iota)_* \, T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right) = \sum_{\boldsymbol{k}^+} (-y)^{|\boldsymbol{k}^+| - |\boldsymbol{k}'|} \, \iota_* \, T_y\left(W_{\boldsymbol{\tau}^+}\right)$$

where $\boldsymbol{\tau}^+ = \left(\boldsymbol{k}^+, \boldsymbol{p}', \boldsymbol{q}'\right)$ and the sum is over the set of weakly increasing sequences $\boldsymbol{k}^+ \geq \boldsymbol{k}' = (1, \dots, k_t)$.

5.1. **The stratification.** The degeneracy locus $W_{\boldsymbol{\tau}}$ is stratified by the loci $W^\circ_{\boldsymbol{\tau}^+} \subseteq W_{\boldsymbol{\tau}}$ defined in (3) with $\boldsymbol{\tau}^+ = (\boldsymbol{k}^+, \boldsymbol{p}', \boldsymbol{q}')$, for weakly increasing sequences $\boldsymbol{k}^+ \geq \boldsymbol{k}'$. After §1.1.3, we set $W_{\boldsymbol{\tau}^+}$ empty, unless $\boldsymbol{\lambda}_{\boldsymbol{\tau}^+}$ is weakly decreasing. The codimension of $W_{\boldsymbol{\tau}^+}$ in $W_{\boldsymbol{\tau}}$ is $|\boldsymbol{\lambda}_{\boldsymbol{\tau}^+}| - |\boldsymbol{\lambda}_{\boldsymbol{\tau}}|$. There is no such stratum of codimension one in $W_{\boldsymbol{\tau}}$.

Recall the variety $\widetilde{\Omega}_{\boldsymbol{\tau}}$ from §3, with the following diagram

$$\begin{array}{ccc} \widetilde{\Omega}_{\boldsymbol{\tau}} & \xhookrightarrow{\ \iota\ } & X_{k_t} \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ W_{\boldsymbol{\tau}} & \hookrightarrow & X. \end{array}$$

The fiber of ϕ over a point x in $W_{\boldsymbol{\tau}}$ is

$$\left\{ (V_1 \subset \cdots \subset V_{k_t}) \in \pi^{-1}(x) \,:\, V_i \subseteq \ker\left(E_{p'_i} \to F_{q'_i}\right)\Big|_x \text{ for all } i \right\}.$$

The map ϕ is locally trivial on each locally closed stratum $W^\circ_{\boldsymbol{\tau}^+} \subseteq W_{\boldsymbol{\tau}}$.

Proof of Theorem 5.1. The collection of loci $W^\circ_{\boldsymbol{\tau}^+}$ from (3), with $\boldsymbol{\tau}^+ = (\boldsymbol{k}^+, \boldsymbol{p}', \boldsymbol{q}')$ for $\boldsymbol{k}^+ \geq \boldsymbol{k}'$, gives a stratification of the locus $W_{\boldsymbol{\tau}}$ such that ϕ is locally trivial on each $W^\circ_{\boldsymbol{\tau}^+}$. Given $\boldsymbol{k}^+ \geq \boldsymbol{k}'$, the generic fiber of ϕ on $W_{\boldsymbol{\tau}^+}$ is isomorphic to the Schubert variety associated to a partition $\boldsymbol{\nu}^+$ assigned to $\boldsymbol{k}^+$:

$$S_{\boldsymbol{\nu}^+} := \left\{ (V_1 \subseteq \cdots \subseteq V_{k_t}) \in \mathrm{Fl}\,(1, \ldots, k_t; \mathbb{C}^{p_t}) \,:\, V_i \subseteq K_{k_i^+} \text{ for all } i \right\},$$

where $K_0 \subset \cdots \subset K_{p_t}$ is a fixed complete flag of vector spaces inside $\mathbb{C}^{p_t}$ with $\dim(K_i) = i$. The vector spaces $K_{k_i^+}$ are meant to be identified with $\ker(E_{p'_i} \to F_{q'_i})\big|_x$ for a generic point x in $W_{\boldsymbol{\tau}^+}$. If $\boldsymbol{k}^+$ is strictly increasing, the partition $\boldsymbol{\nu}^+$ is defined as

$$\boldsymbol{\nu}^+ := (k^+_{k_t} - k_t, \ldots, k^+_1 - 1).$$

In general, for $\boldsymbol{k}^+$ not necessarily strictly increasing, a coordinate computation shows that the Schubert cell $S^\circ_{\boldsymbol{\nu}^+}$ in $S_{\boldsymbol{\nu}^+}$ is isomorphic to the affine space $\mathbb{A}^{|\boldsymbol{k}^+| - |\boldsymbol{k}'|}$.

As in Lemma 2.4, one has

$$(\pi\iota)_*\, T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right) = \sum_{\boldsymbol{k}^+} d_{(\boldsymbol{k}^+)}\, \iota_*\, T_y\left(W_{\boldsymbol{\tau}^+}\right)$$

with $d_{(\boldsymbol{k}^+)} = \chi_y\left(S_{\boldsymbol{\nu}^+}\right) - \sum_{\boldsymbol{f}} d_{\boldsymbol{f}}$, where the sum is over the set of weakly increasing sequences $\boldsymbol{f}$ such that $W_{\boldsymbol{\phi}} \supset W_{\boldsymbol{\tau}^+}$ for $\boldsymbol{\phi} = (\boldsymbol{f}, \boldsymbol{p}', \boldsymbol{q}')$, that is, $\boldsymbol{f} < \boldsymbol{k}^+$ and $\boldsymbol{\lambda}_{\boldsymbol{\phi}}$ is weakly decreasing. Clearly $d_{(\boldsymbol{k}')} = 1$, and by recursion one finds that $d_{(\boldsymbol{k}^+)}$ is equal to the Hirzebruch χ_y-genus of the Schubert cell $(S_{\boldsymbol{\nu}^+})^\circ$. Since $(S_{\boldsymbol{\nu}^+})^\circ \cong \mathbb{A}^{|\boldsymbol{k}^+| - |\boldsymbol{k}'|}$, one has $d_{(\boldsymbol{k}^+)} = (-y)^{|\boldsymbol{k}^+| - |\boldsymbol{k}'|}$ for each $\boldsymbol{k}^+$ (as in Example 2.5), hence the statement. □

Example 5.2. Consider the triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ where $\boldsymbol{p} = (2, 3)$, $\boldsymbol{q} = (3, 2)$, and $\boldsymbol{k} = (1, 2)$. One has

$$(\pi\iota)_*\, T_y\left(\widetilde{\Omega}_{\boldsymbol{\tau}}\right) = \iota_*\, T_y\left(W_{(1,2)}\right) - y\, \iota_*\, T_y\left(W_{(2,2)}\right) - y\, \iota_*\, T_y\left(W_{(1,3)}\right) + y^2\, \iota_*\, T_y\left(W_{(2,3)}\right).$$

Here for simplicity, we use the notation $W_{\boldsymbol{k}^+} := W_{\boldsymbol{\tau}^+}$. For each stratum, one has the following configuration over its general point:

$$\begin{aligned}
W_{(1,2)}: &\quad \dim \mathrm{Ker}(E_2 \to F_3) = 1 \quad \text{and} \quad \dim \mathrm{Ker}(E_3 \to F_2) = 2\\
W_{(2,2)}: &\quad \dim \mathrm{Ker}(E_2 \to F_3) = 2 \quad \text{and} \quad \dim \mathrm{Ker}(E_3 \to F_2) = 2\\
W_{(1,3)}: &\quad \dim \mathrm{Ker}(E_2 \to F_3) = 1 \quad \text{and} \quad \dim \mathrm{Ker}(E_3 \to F_2) = 3\\
W_{(2,3)}: &\quad \dim \mathrm{Ker}(E_2 \to F_3) = 2 \quad \text{and} \quad \dim \mathrm{Ker}(E_3 \to F_2) = 3.
\end{aligned}$$

5.2. **Example: Schubert varieties in Grassmannians.** The Schubert varieties of a Grassmannian are *Grassmannian degeneracy loci* for maps from the tautological vector bundle to a flag of constant bundles. Our results apply to give the motivic Hirzebruch class of the Schubert varieties in terms of the motivic Hirzebruch class of the Grassmannian. For instance, for $X = G_2\left(\mathbb{C}^5\right)$, consider the Schubert variety $S_{\boldsymbol{\lambda}}$ associated to the partition $\boldsymbol{\lambda} = (2,1)$. This is the degeneracy locus $W_{\boldsymbol{\tau}}$ corresponding to the triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ with $\boldsymbol{k} = (1,2)$, $\boldsymbol{p} = (2,2)$, and $\boldsymbol{q} = (3,1)$. The stratification here consists of two strata: the stratum with $\boldsymbol{k}^+ = (1,2)$ and the one with $\boldsymbol{k}^+ = (2,2)$. For $y = -1$, Theorems 5.1 and 3.1 give

$$\begin{aligned}
\iota_*\, c_{\mathrm{SM}}\left(S_{\boldsymbol{\lambda}}\right) &= \left(\ydiagram{2,1} - 2\ydiagram{3,1} - 2\ydiagram{2,2} + 5\ydiagram{3,2} - 4\ydiagram{3,3}\right) \cap c_{\mathrm{SM}}(X)\\
&= \ydiagram{2,1} + 3\ydiagram{3,1} + 3\ydiagram{2,2} + 8\ydiagram{3,2} + 5\ydiagram{3,3}.
\end{aligned}$$

The last equality uses the formula for $c_{\mathrm{SM}}(X)$ from [1], and the resulting formula checks with the CSM class computation in [1].

6. The locus $\Omega_{\boldsymbol{\lambda}}$ and its motivic Hirzebruch class

We discuss here a setting which is particularly relevant in the study of pointed Brill-Noether varieties (§7). Consider the following maps of vector bundles over a variety X:

$$E_p \xrightarrow{\varphi} F_{q_1} \twoheadrightarrow F_{q_2} \cdots \twoheadrightarrow F_{q_t}.$$

The Grassmann degeneracy locus is

$$W_{\boldsymbol{\lambda}} := \left\{x \in X \,:\, \dim \ker(E_p \to F_{q_i})|_x \geq i \ \text{ for all } i\right\},$$

with partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t)$ from §1.2 equal to $\lambda_i := q_i - p + i$. The locus $W_{\boldsymbol{\lambda}}$ is the degeneracy locus $W_{\boldsymbol{\tau}}$ corresponding to the triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$ with $\boldsymbol{k} = (1, \ldots, t)$, $\boldsymbol{p} = (p, \ldots, p)$, and $\boldsymbol{q} = (q_1, \ldots, q_t)$. Its motivic Hirzebruch class is computed by Theorems 5.1 and 3.1. Now consider the Grassmannian bundle $\mathbf{Gr}\left(t, E_p\right)$ on X with tautological rank t sub-bundle $\mathbb{S}$, and define its subvariety $\Omega_{\boldsymbol{\lambda}}$ by the conditions

$$\dim \ker(\mathbb{S} \to F_{q_i}) \geq i \quad \text{for } 1 \leq i \leq t.$$

One has

$$\begin{array}{ccc}
\Omega_{\boldsymbol{\lambda}} & \xhookrightarrow{\ \iota\ } & \mathbf{Gr}(t, E_p)\\
{\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\pi}\\
W_{\boldsymbol{\lambda}} & \hookrightarrow & X,
\end{array}$$

and the fiber of ϕ over a point x is

$$\{V \in \mathrm{Gr}\,(t, E_p|_x) \,:\, \dim\left(V \cap \ker\left(E_p \to F_{q_i}\right)|_x\right) \geq i \text{ for all } i\}\,. \tag{14}$$

The aim of this section is to describe the (push-forward of the) motivic Hirzebruch class of $\Omega_{\boldsymbol{\lambda}}$ in terms of the motivic Hirzebruch class of loci of type $W_{\boldsymbol{\lambda}}$. This is achieved in Proposition 6.4. The (push-forward of the) motivic Hirzebruch class of $\Omega_{\boldsymbol{\lambda}}$ then follows after applying Theorems 5.1 and 3.1 to compute the motivic Hirzebruch class of the loci $W_{\boldsymbol{\lambda}}$.

6.1. **The stratification.** We start by describing a stratification of $W_{\boldsymbol{\lambda}}$ induced by the projection $\phi\colon \Omega_{\boldsymbol{\lambda}} \to W_{\boldsymbol{\lambda}}$. Consider the loci

$$W^{\boldsymbol{\kappa}}_{\boldsymbol{\lambda}} := \{x \in X \,:\, \dim\ker\left(E_p \to F_{q_i}\right)|_x \geq i + \kappa_i \text{ for all } i\} \subseteq W_{\boldsymbol{\lambda}},$$

with $\boldsymbol{\kappa} = (0 \leq \kappa_1 \leq \cdots \leq \kappa_t)$ such that $\boldsymbol{\lambda} + \boldsymbol{\kappa}$ is a partition. The map ϕ is locally trivial precisely on the locally closed strata

$$(W^{\boldsymbol{\kappa}}_{\boldsymbol{\lambda}})^\circ := \{x \in X \,:\, \dim\ker\left(E_p \to F_{q_i}\right)|_x = i + \kappa_i \text{ for all } i\} \subseteq W^{\boldsymbol{\kappa}}_{\boldsymbol{\lambda}}.$$

The fiber of ϕ over a general point x in $W^{\boldsymbol{\kappa}}_{\boldsymbol{\lambda}}$ from (14) coincides with the Schubert variety $S_{\boldsymbol{\kappa}^c}$ in $\mathrm{Gr}(t, E_p|_x)$ associated with the partition $\boldsymbol{\kappa}^c$ complementary to $\boldsymbol{\kappa}$ inside the $t \times (p-t)$ rectangle. For instance, when $t = 3$, $p = 7$, and $\boldsymbol{\kappa} = (1, 1, 2)$, one has $\boldsymbol{\kappa}^c = (3, 3, 2)$.

In the following, it is convenient to identify a weakly increasing sequence $\boldsymbol{\kappa}$ with the shape consisting of κ_i boxes in the ith row, and note that the componentwise order is compatible with containment of shapes.

Remark 6.1. Fix $\boldsymbol{\kappa} = (0 \leq \kappa_1 \leq \cdots \leq \kappa_t)$ such that $\boldsymbol{\lambda} + \boldsymbol{\kappa}$ is a partition. We emphasize that the partition in §1.2 assigned to the degeneracy locus $W^{\boldsymbol{\kappa}}_{\boldsymbol{\lambda}}$ is not $\boldsymbol{\lambda} + \boldsymbol{\kappa}$, but rather the partition with $1 + \kappa_1$ parts equal to $\lambda_1 + \kappa_1$, and $1 + \kappa_2 - \kappa_1$ parts equal to $\lambda_2 + \kappa_2$, etc.

6.2. **The sequence $\boldsymbol{\kappa}^{\mathrm{red}}$.** Fix a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t)$ and $\boldsymbol{\kappa} = (0 \leq \kappa_1 \leq \cdots \leq \kappa_t)$ such that $\boldsymbol{\lambda} + \boldsymbol{\kappa}$ is weakly decreasing.

Definition 6.2. Given $\boldsymbol{\lambda}$ and $\boldsymbol{\kappa}$ as above, the sequence

$$\boldsymbol{\kappa}^{\mathrm{red}} = (0 \leq \kappa_1^{\mathrm{red}} \leq \cdots \leq \kappa_t^{\mathrm{red}})$$

is defined as the minimal sequence in componentwise order with $\kappa_t^{\mathrm{red}} = \kappa_t$ and

$$\kappa_i^{\mathrm{red}} = \kappa_i \qquad \text{when } \kappa_{i+1} + \lambda_{i+1} < \kappa_i + \lambda_i, \text{ for } i < t.$$

For instance, for $\boldsymbol{\lambda} = (4, 4, 1, 1)$ and $\boldsymbol{\kappa} = (1, 1, 3, 3)$ one has $\boldsymbol{\kappa}^{\mathrm{red}} = (0, 1, 1, 3)$. See Example 6.7 for a graphical representation of some sequences $\boldsymbol{\kappa}$ and corresponding $\boldsymbol{\kappa}^{\mathrm{red}}$.

Lemma 6.3. *Given $\boldsymbol{\lambda}$ and $\boldsymbol{\kappa}$ as above, $\boldsymbol{\kappa}^{\mathrm{red}}$ is the smallest shape inside $\boldsymbol{\kappa}$ which is not contained in any of the shapes $\boldsymbol{\epsilon} < \boldsymbol{\kappa}$ such that $\boldsymbol{\lambda} + \boldsymbol{\epsilon}$ is weakly decreasing.*

Proof. We first prove that if $\boldsymbol{\epsilon}$ contains $\boldsymbol{\kappa}^{\mathrm{red}}$ and $\boldsymbol{\lambda} + \boldsymbol{\epsilon}$ is weakly decreasing, then $\boldsymbol{\epsilon} = \boldsymbol{\kappa}$. One has necessarily $\epsilon_t = \kappa_t$. Next, fix $i < t$ such that $\kappa_i^{\mathrm{red}} < \kappa_i$, and assume that we have already settled that $\epsilon_j = \kappa_j$ for $j > i$. By definition, since $\kappa_i^{\mathrm{red}} < \kappa_i$, one has $\kappa_{i+1} + \lambda_{i+1} = \kappa_i + \lambda_i$. Since we want that $\lambda_i + \epsilon_i \geq \lambda_{i+1} + \epsilon_{i+1} = \lambda_{i+1} + \kappa_{i+1}$, then necessarily $\epsilon_i = \kappa_i$.

Finally, we show that any smaller shape $\boldsymbol{\kappa}' < \boldsymbol{\kappa}^{\mathrm{red}}$ is contained inside some $\boldsymbol{\epsilon} < \boldsymbol{\kappa}$ such that $\boldsymbol{\lambda} + \boldsymbol{\epsilon}$ is weakly decreasing. If $\kappa'_t < \kappa_t = \kappa_t^{\mathrm{red}}$, then $\boldsymbol{\kappa}'$ is contained inside

$$\boldsymbol{\epsilon} = (\min\{\kappa_1, \kappa_t - 1\}, \ldots, \min\{\kappa_{t-1}, \kappa_t - 1\}, \kappa_t - 1).$$

Next, fix $i < t$ such that $\kappa_i^{\mathrm{red}} > \kappa_{i-1}^{\mathrm{red}}$, or $i = 1$ and $\kappa_1 > 0$. By the definition of $\boldsymbol{\kappa}^{\mathrm{red}}$, this implies that $\kappa_i^{\mathrm{red}} = \kappa_i$ and $\kappa_{i+1} + \lambda_{i+1} < \kappa_i + \lambda_i$. Consider $\boldsymbol{\kappa}'$ defined as $\kappa'_i = \kappa_i^{\mathrm{red}} - 1$, and $\kappa'_j = \kappa_j^{\mathrm{red}}$ for $j \neq i$. Then $\boldsymbol{\kappa}'$ is contained inside

$$\boldsymbol{\epsilon} = (\min\{\kappa_1, \kappa_i - 1\}, \ldots, \min\{\kappa_{i-1}, \kappa_i - 1\}, \kappa_i - 1, \kappa_{i+1}, \ldots, \kappa_t).$$

The condition $\kappa_{i+1} + \lambda_{i+1} < \kappa_i + \lambda_i$ guarantees that $\boldsymbol{\lambda} + \boldsymbol{\epsilon}$ is weakly decreasing. □

6.3. **The motivic Hirzebruch class of $\Omega_{\boldsymbol{\lambda}}$.**

Proposition 6.4. *With assumptions as in §1.4, one has*

$$(\pi\iota)_* T_y(\Omega_{\boldsymbol{\lambda}}) = \sum_{\boldsymbol{\kappa} \geq 0} d_{\boldsymbol{\kappa}}\, \iota_* T_y(W_{\boldsymbol{\lambda}}^{\boldsymbol{\kappa}})$$

where the sum is over $\boldsymbol{\kappa} = (0 \leq \kappa_1 \leq \cdots \leq \kappa_t)$ such that $\boldsymbol{\lambda} + \boldsymbol{\kappa}$ is weakly decreasing and

$$d_{\boldsymbol{\kappa}} := \sum_{\substack{\boldsymbol{\kappa}' = (\kappa'_1 \leq \cdots \leq \kappa'_t) \\ \boldsymbol{\kappa}^{\mathrm{red}} \leq \boldsymbol{\kappa}' \leq \boldsymbol{\kappa}}} (-y)^{|\boldsymbol{\kappa}'|}. \tag{15}$$

Example 6.5. When $y = -1$, the topological Euler characteristic of the fibers (14) of ϕ and consequently the coefficients $d_{\boldsymbol{\kappa}}$ are computed as follows.

For $\boldsymbol{\kappa} = (0 \leq \kappa_1 \leq \cdots \leq \kappa_l)$, the topological Euler characteristic of the Schubert variety $S_{\boldsymbol{\kappa}^c}$ in $\mathrm{Gr}(l, \mathbb{C}^p)$ associated with the partition complementary to $\boldsymbol{\kappa}$ inside the $l \times (p-l)$ rectangle can be computed as

$$\chi(S_{\boldsymbol{\kappa}^c}) = p(\boldsymbol{\kappa}) := \left| \binom{\kappa_{l+1-j} + l - i + 1}{1 + j - i} \right|_{1 \leq i,j \leq l}. \tag{16}$$

Indeed, $\chi(S_{\boldsymbol{\kappa}^c})$ coincides with the number of partitions inside $(\kappa_l, \ldots, \kappa_1)$, or equivalently, the number of shapes inside the shape $\boldsymbol{\kappa}$. This equals the number of non-intersecting lattice paths

$$\begin{aligned} \text{from the points} \quad & (0,0), (1,0), \ldots, (l-1, 0) \\ \text{to the points} \quad & (1, \kappa_l + l - 1), (2, \kappa_{l-1} + l - 2), \ldots, (l, \kappa_1) \end{aligned}$$

(see e.g., [26]). From the Gessel-Viennot formula [23], this number equals the above $p(\boldsymbol{\kappa})$, hence the first equality in (16).

By definition (15), when $y = -1$ the coefficient $d_{\boldsymbol{\kappa}}$ counts the number of shapes inside the shape $\boldsymbol{\kappa}$ and containing the shape $\boldsymbol{\kappa}^{\mathrm{red}}$, that is, the number of shapes in $\boldsymbol{\kappa} \smallsetminus \boldsymbol{\kappa}^{\mathrm{red}}$. Now Lemma 6.3 implies that $\boldsymbol{\kappa} \smallsetminus \boldsymbol{\kappa}^{\mathrm{red}}$ is a disjoint union of shorter weakly increasing sequences, say $\boldsymbol{\kappa}^{(1)}, \boldsymbol{\kappa}^{(2)}, \ldots$. The number of shapes in $\boldsymbol{\kappa} \smallsetminus \boldsymbol{\kappa}^{\mathrm{red}}$ is equal to the product of the number of shapes inside each $\boldsymbol{\kappa}^{(i)}$. It follows that $d_{\boldsymbol{\kappa}}$ can be computed as the product of the quantities $p\left(\boldsymbol{\kappa}^{(i)}\right)$.

For instance, when $\boldsymbol{\lambda} = (4, 4, 1, 1)$ and $\boldsymbol{\kappa} = (1, 1, 3, 3)$, one has $\boldsymbol{\kappa}^{\mathrm{red}} = (0, 1, 1, 3)$, hence $\boldsymbol{\kappa} \smallsetminus \boldsymbol{\kappa}^{\mathrm{red}}$ is the disjoint union of the length one sequences (1) and (2) (see also how the

white tiles representing $\boldsymbol{\kappa} \smallsetminus \boldsymbol{\kappa}^{\mathrm{red}}$ are indeed union of disjoint shapes in Example 6.7) and indeed $d_{\boldsymbol{\kappa}} = p((1)) \cdot p((2)) = 2 \cdot 3 = 6$.

Example 6.6. When $\lambda_1 = \cdots = \lambda_t$, one has necessarily $\kappa_1 = \cdots = \kappa_t$, hence we have $\boldsymbol{\kappa} = (\kappa, \ldots, \kappa)$. For $\boldsymbol{\kappa} = (\kappa, \ldots, \kappa) =: \kappa^t$, one has $\boldsymbol{\kappa}^{\mathrm{red}} = (0, \ldots, 0, \kappa)$ and thus $\boldsymbol{\kappa} \smallsetminus \boldsymbol{\kappa}^{\mathrm{red}} \cong \kappa^{t-1}$. For $y = -t^2$, the right-hand side of (15) is $t^{2|\boldsymbol{\kappa}^{\mathrm{red}}|} P_{(\kappa, t-1+\kappa)}(t)$, hence we have

$$d_{\boldsymbol{\kappa}} = t^{2\kappa} P_{(\kappa, t-1+\kappa)}(t),$$

where $P_{(\kappa,t-1+\kappa)}(t)$ is the Poincaré polynomial of the Grassmannian $\mathrm{Gr}(\kappa, \mathbb{C}^{t-1+\kappa})$. Since $P_{(\kappa,t-1+\kappa)}(t)$ is given by the t^2-binomial coefficient

$$P_{(\kappa,t-1+\kappa)}(t) = \left[\begin{array}{c} t-1+\kappa \\ \kappa \end{array} \right]_{t^2} := \frac{\prod_{i=1}^{t-1+\kappa} \left(1 - t^{2i}\right)}{\prod_{i=1}^{\kappa} \left(1 - t^{2i}\right) \prod_{i=1}^{t-1} \left(1 - t^{2i}\right)},$$

we have

$$(\pi\iota)_* \, T_y \left(\Omega_{\boldsymbol{\lambda}}\right) = \sum_{\kappa \geq 0} (-y)^{\kappa} \left[\begin{array}{c} t-1+\kappa \\ \kappa \end{array} \right]_{-y} \iota_* \, T_y \left(W_{\boldsymbol{\lambda}}^{\boldsymbol{\kappa}}\right)$$

where $\boldsymbol{\kappa} = \kappa^t$. For $y = -1$, the coefficients in the sum specialize to the binomial coefficients $\binom{t-1+\kappa}{\kappa}$.

Example 6.7. When $\boldsymbol{\lambda} = (4, 4, 1, 1)$, one has that $(\pi\iota)_* \, T_y \left(\Omega_{\boldsymbol{\lambda}}\right)$ equals

$$1 - y(1-y) + y^2\left(1 - y + y^2\right) - y^3\left(1 - y + y^2 - y^3\right)$$
$$- y^3(1-y) + y^4(1-y)^2 - y^5\left(1 - y + y^2\right)(1-y)$$
$$+ y^4\left(1 - y + 2y^2 - 3y^3 + 3y^4 - 2y^5 + y^6\right)$$
$$+ y^6\left(1 - y + y^2\right) - y^7(1-y)\left(1 - y + y^2\right)$$
$$+ y^8\left(1 - y + y^2\right)^2$$
$$- y^5\left(1 - y + 2y^2 - 3y^3 + 4y^4 - 5y^5 + 5y^6 - 4y^7 + 2y^8 - y^9\right) \cdots .$$

Here for simplicity, the shape $\boldsymbol{\kappa}$ with κ_i boxes in the ith row stands for $\iota_* \, T_y \left(W_{\boldsymbol{\lambda}}^{\boldsymbol{\kappa}}\right)$. Inside each $\boldsymbol{\kappa}$, the shape $\boldsymbol{\kappa}^{\mathrm{red}}$ is shaded. Note how the complement of $\boldsymbol{\kappa}^{\mathrm{red}}$ in each $\boldsymbol{\kappa}$ is a disjoint union of shapes.

Proof of Proposition 6.4. Write

$$(\pi\iota)_* T_y \left(\Omega_{\boldsymbol{\lambda}}\right) = \sum_{\boldsymbol{\kappa} \geq 0} d_{\boldsymbol{\kappa}} \, \iota_* \, T_y \left(W_{\boldsymbol{\lambda}}^{\boldsymbol{\kappa}}\right)$$

for some coefficients $d_{\boldsymbol{\kappa}}$. One has $W^{\boldsymbol{\epsilon}}_{\boldsymbol{\lambda}} \supset W^{\boldsymbol{\kappa}}_{\boldsymbol{\lambda}}$ exactly when $\boldsymbol{\epsilon}$ is contained inside $\boldsymbol{\kappa}$, that is, $\epsilon_i \leq \kappa_i$ for all i. From §6.1, the fiber of ϕ over a general point in $W^{\boldsymbol{\kappa}}_{\boldsymbol{\lambda}}$ is the Schubert variety $S_{\boldsymbol{\kappa}^c}$ in $\mathrm{Gr}(t, \mathbb{C}^p)$ associated with the partition $\boldsymbol{\kappa}^c$ complementary to $\boldsymbol{\kappa}$ inside the $t \times (p-t)$ rectangle. Hence from Lemma 2.4, we have

$$d_{\boldsymbol{\kappa}} = \chi_y(S_{\boldsymbol{\kappa}^c}) - \sum_{\boldsymbol{\epsilon} < \boldsymbol{\kappa}} d_{\boldsymbol{\epsilon}}$$

where the sum is over $\boldsymbol{\epsilon}$ such that $\boldsymbol{\lambda} + \boldsymbol{\epsilon}$ is weakly decreasing, and $\epsilon_i \leq \kappa_i$ for all i, with at least one strict inequality.

The Schubert variety $S_{\boldsymbol{\kappa}^c}$ is the union of the Schubert cells corresponding to (weakly decreasing) partitions contained inside $(\kappa_t, \ldots, \kappa_1)$, or equivalently, shapes inside the shape $\boldsymbol{\kappa}$. The sum

$$\sum_{\boldsymbol{\epsilon} < \boldsymbol{\kappa}} d_{\boldsymbol{\epsilon}}$$

is the sum of the Hirzebruch χ_y-genera of Schubert cells corresponding to shapes contained inside some $\boldsymbol{\epsilon} < \boldsymbol{\kappa}$ such that $\boldsymbol{\lambda} + \boldsymbol{\epsilon}$ is weakly decreasing. From Lemma 6.3, it follows that $d_{\boldsymbol{\kappa}}$ is the sum of the Hirzebruch χ_y-genera of Schubert cells corresponding to shapes contained inside $\boldsymbol{\kappa}$ and containing $\boldsymbol{\kappa}^{\mathrm{red}}$. Since a Schubert cell corresponding to a shape $\boldsymbol{\kappa}'$ is isomorphic to the affine space $\mathbb{A}^{|\boldsymbol{\kappa}'|}_{\mathbb{C}}$, its Hirzebruch χ_y-genus is $(-y)^{|\boldsymbol{\kappa}'|}$ (Example 2.5). The statement follows. □

7. Pointed Brill-Noether varieties

For a smooth pointed curve (C, P) and a sequence $\boldsymbol{a} : 0 \leq a_0 < \cdots < a_r \leq d$, the *pointed Brill-Noether variety of line bundles* $W^{\boldsymbol{a}}_d(C, P)$ is defined as

$$W^{\boldsymbol{a}}_d(C, P) := \left\{ L \in \mathrm{Pic}^d(C) \,|\, h^0(C, L \otimes \mathscr{O}_C(-a_i P)) \geq r + 1 - i \text{ for all } i \right\}.$$

The *pointed Brill-Noether variety of linear series* $G^{\boldsymbol{a}}_d(C, P)$ is defined as

$$G^{\boldsymbol{a}}_d(C, P) := \left\{ (L, V) \,\middle|\, \begin{array}{l} L \in \mathrm{Pic}^d(C),\, V \subseteq H^0(C, L),\, \dim(V) = r+1, \text{ and} \\ \dim\left(V \cap H^0(C, L \otimes \mathscr{O}_C(-a_i P))\right) \geq r + 1 - i \text{ for all } i \end{array} \right\}.$$

The variety $W^{\boldsymbol{a}}_d(C, P)$ has the structure of a Grassmannian degeneracy locus in $\mathrm{Pic}^d(C)$, and the variety $G^{\boldsymbol{a}}_d(C, P)$ is of type $\Omega_{\boldsymbol{\lambda}}$, as in §6. We briefly review this in §7.1, we verify the assumptions from §1.4 in §7.2, and apply Theorems 5.1 and 3.1 to compute the motivic Hirzebruch class of pointed Brill-Noether varieties in §7.3. Finally, we conclude with some examples.

7.1. **The construction.** Choose a positive integer n large enough so that line bundles of degree $d+n$ are non-special, that is, $n \geq 2g - 1 - d$. Fix a Poincaré line bundle $\mathscr{L}$ on $C \times \mathrm{Pic}^d(C)$, normalized so that $\mathscr{L}|_{\{P\} \times \mathrm{Pic}^d(C)}$ is trivial. Consider the following vector bundles on $\mathrm{Pic}^d(C)$:

$$\begin{aligned} \mathscr{E} &:= (\pi_2)_* \left(\mathscr{L} \otimes \pi_1^* \mathscr{O}_C(nP) \right), \\ \mathscr{F}_i &:= (\pi_2)_* \left(\mathscr{L} \otimes \pi_1^* \mathscr{O}_{(n + a_{r+1-i})P} \right) \qquad \text{for } 1 \leq i \leq r+1. \end{aligned}$$

Here π_1 and π_2 are the projections from $C \times \mathrm{Pic}^d(C)$ to C and $\mathrm{Pic}^d(C)$, respectively. One computes

$$\begin{aligned} p &:= \mathrm{rank}\,(\mathscr{E}) = d + n - g + 1, \\ q_i &:= \mathrm{rank}\,(\mathscr{F}_i) = n + a_{r+1-i} \qquad \text{for } 1 \leq i \leq r+1. \end{aligned} \tag{17}$$

There are natural maps

$$\mathscr{E} \to \mathscr{F}_1 \twoheadrightarrow \mathscr{F}_2 \cdots \twoheadrightarrow \mathscr{F}_{r+1}$$

and $W^{\boldsymbol{a}}_d(C,P)$ is the Grassmannian degeneracy locus with partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{r+1})$ from §1.2 equal to

$$\lambda_i := g - d + r + a_{r+1-i} - (r+1-i) \qquad \text{for } 1 \leq i \leq r+1. \tag{18}$$

One has $c(\mathscr{F}_i) = 0$ for all i, and all classes $c(i)$ from §1.3 are equal to $c = c(-\mathscr{E}) = \mathrm{e}^{\theta}$ in $H^*(\mathrm{Pic}^d(C))$, where θ is the cohomology class of the theta divisor [9, §VIII]. Finally, note that $T_y\left(\mathrm{Pic}^d(C)\right) = 1$, as Abelian varieties have trivial tangent bundles.

7.2. **Dimension and singular locus.** The *one-pointed Brill-Noether Theorem* [15, §1] says that for a *general* smooth pointed curve (C,P): (i) the varieties $W^{\boldsymbol{a}}_d(C,P)$ and $G^{\boldsymbol{a}}_d(C,P)$ are non-empty if and only if $g \geq \sum_{i=0}^{r} \max\{0, g-d+r+a_i-i\}$, and (ii) when non-empty, $G^{\boldsymbol{a}}_d(C,P)$ has dimension equal to the *one-pointed Brill-Noether number* $\rho(g,r,d,\boldsymbol{a}) := g - \sum_{i=0}^{r}(g-d+r+a_i-i) = g - |\boldsymbol{\lambda}|$; the same holds for $W^{\boldsymbol{a}}_d(C,P)$ when $\rho(g,r,d,\boldsymbol{a}) \leq g$.

The proof in [15] uses degenerations to singular curves. However, explicit examples of *smooth* pointed curves of any genus verifying the one-pointed Brill-Noether Theorem and defined over $\mathbb{Q}$ were provided in [16].

Furthermore, the *one-pointed Gieseker-Petri Theorem* [13] characterizes the smooth locus of $G^{\boldsymbol{a}}_d(C,P)$ for a *general* smooth pointed curve (C,P), and implies that the singular locus of $G^{\boldsymbol{a}}_d(C,P)$ is contained in the locus of linear series with excess vanishing at P. Since for $\rho(g,r,d,\boldsymbol{a}) \leq g$ the forgetful map $\pi\colon G^{\boldsymbol{a}}_d(C,P) \to W^{\boldsymbol{a}}_d(C,P)$ is an isomorphism when restricted over the locus

$$W^{\boldsymbol{a}}_d(C,P)^{\circ} := \left\{ L \in \mathrm{Pic}^d(C) \,|\, h^0(C, L \otimes \mathscr{O}_C(-a_i P)) = r+1-i \text{ for all } i \right\} \subseteq W^{\boldsymbol{a}}_d(C,P)$$

and the smooth locus of $G^{\boldsymbol{a}}_d(C,P)$ as described in [13] includes $\pi^{-1}\left(W^{\boldsymbol{a}}_d(C,P)^{\circ}\right)$, it follows that $W^{\boldsymbol{a}}_d(C,P)^{\circ}$ is smooth for a general (C,P) and $\rho(g,r,d,\boldsymbol{a}) \leq g$.

The stratification of $W^{\boldsymbol{a}}_d(C,P)$ from §5.1 consists of strata whose closures are themselves pointed Brill-Noether varieties of line bundles inside $\mathrm{Pic}^d(C)$. Namely, the strata are

$$W^{\boldsymbol{a}^+}_d(C,P) := \left\{ L \in \mathrm{Pic}^d(C) \,|\, h^0(C, L \otimes \mathscr{O}_C(-a_i P)) \geq k^+_{r+1-i} \text{ for all } i \right\} \tag{19}$$

for $\boldsymbol{k}^+ \geq \boldsymbol{k} = (1, \ldots, r+1)$. (Note that some of the conditions defining $W^{\boldsymbol{a}^+}_d(C,P)$ may be redundant.)

In particular, $W^{\boldsymbol{a}}_d(C,P)$ satisfies the transversality assumption from §1.4 for a general (C,P) and $\rho(g,r,d,\boldsymbol{a}) \leq g$.

While in this paper we only treat the one-pointed case, these results are also known more generally for two-pointed Brill-Noether varieties with appropriate changes, see [15, 33, 13].

7.3. **Motivic Hirzebruch class of pointed Brill-Noether varieties.** For the pointed Brill-Noether variety $W_d^{\boldsymbol{a}}(C,P)$ with $\rho(g,r,d,\boldsymbol{a}) \leq g$, the (push-forward of the) motivic Hirzebruch class of its resolution from Theorem 3.1 equals

$$A_{\boldsymbol{\lambda}} := \prod_{i=1}^{r+1} \frac{\prod_{k=1}^{p} Q_y(\alpha_k + R_i)}{\prod_{j=1}^{i-1} Q_y(R_i - R_j)} \left(Q_y(R_i)\right)^{-q_i} \left|c_{\lambda_i+j-i}\right|_{1\leq i,j\leq r+1} \tag{20}$$

where $\alpha_1, \ldots, \alpha_p$ are the Chern roots of the vector bundle $\mathscr{E}$. These satisfy

$$\sum_{k=1}^{p} \alpha_k = -\theta \qquad \text{and} \qquad \sum_{k=1}^{p} \alpha_k^i = 0 \quad \text{for } i > 1 \tag{21}$$

[9, pg. 336]. The expression (20) is symmetric in the α_k, hence after expanding and using (21), can be rewritten as a polynomial in θ with coefficients in $H^*\left(\mathrm{Pic}^d(C)\right)[y]$. For instance, the first terms of $\prod_{k=1}^{p} Q_y(\alpha_k + R_i)$ are

$$Q_y(R_i)^p - \frac{1-y}{2}\,\theta\, Q_y(R_i)^{p-1} - \frac{(1+y)^2}{6}\theta\, R_i\, Q_y(R_i)^{p-1} + \frac{(1-y)^2}{8}\,\theta^2\, Q_y(R_i)^{p-2} + \ldots.$$

When $y = -1$, the expression $A_{\boldsymbol{\lambda}}$ reduces as follows:

$$\prod_{i>j} \frac{1}{1+R_i-R_j} \left| \frac{(1+R_i)^{-\lambda_i+i}}{\mathrm{e}^{\frac{\theta}{1+R_i}}} c_{\lambda_i+j-i} \right|_{1\leq i,j\leq k_{r+1}}.$$

Theorems 5.1 and 3.1 imply:

Corollary 7.1. *For a general smooth pointed curve (C,P) of genus g and for a sequence $\boldsymbol{a}: 0 \leq a_0 < \cdots < a_r \leq d$ such that $\rho(g,r,d,\boldsymbol{a}) \leq g$, one has*

$$A_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{k}^+} (-y)^{|\boldsymbol{k}^+|-|\boldsymbol{k}|}\, \iota_*\, T_y\left(W_d^{\boldsymbol{a}^+}(C,P)\right) \qquad \text{in } H^*\left(\mathrm{Pic}^d(C)\right)[y],$$

where the sum is over the set of weakly increasing sequences $\boldsymbol{k}^+ \geq \boldsymbol{k} = (1,\ldots,r+1)$, and for a given $\boldsymbol{k}^+$, the locus $W_d^{\boldsymbol{a}^+}(C,P) \subseteq W_d^{\boldsymbol{a}}(C,P)$ is as in (19).

By the inclusion-exclusion principle, this determines the class $\iota_*\, T_y\left(W_d^{\boldsymbol{a}}(C,P)\right)$ in $H^*\left(\mathrm{Pic}^d(C)\right)[y]$ in terms of classes of type $A_{\boldsymbol{\lambda}}$.

Similarly, Proposition 6.4 implies:

Corollary 7.2. *For a general smooth pointed curve (C,P) of genus g and for a sequence $\boldsymbol{a}: 0 \leq a_0 < \cdots < a_r \leq d$ such that $\rho(g,r,d,\boldsymbol{a}) \leq g$, one has*

$$(\pi\iota)_*\, T_y\left(G_d^{\boldsymbol{a}}(C,P)\right) = \sum_{\boldsymbol{\kappa}\geq 0} d_{\boldsymbol{\kappa}}\, \iota_*\, T_y\left(W_d^{\boldsymbol{a}(\boldsymbol{\kappa})}(C,P)\right) \qquad \text{in } H^*\left(\mathrm{Pic}^d(C)\right)[y],$$

where the sum is over $\boldsymbol{\kappa} = (0 \leq \kappa_1 \leq \cdots \leq \kappa_{r+1})$ such that $\boldsymbol{\lambda} + \boldsymbol{\kappa}$ is weakly decreasing, the coefficients $d_{\boldsymbol{\kappa}}$ are as in (15)*, and $\boldsymbol{a}(\boldsymbol{\kappa})$ is the strictly increasing sequence*

$$\boldsymbol{a}(\boldsymbol{\kappa}) = \left(a_0, a_0+1, \ldots, a_0+\kappa_{r+1}-\kappa_r, \ldots, a_r, a_r+1, \ldots, a_r+\kappa_1\right).$$

The expression for $\boldsymbol{a}(\boldsymbol{\kappa})$ is obtained from (18) using the partition λ assigned to $W_d^{\boldsymbol{a}(\boldsymbol{\kappa})}(C,P)$ as in Remark 6.1. Examples are studied below.

7.4. **The curve case.** When (C, P) is general of genus $g \geq 1$ and $W^{\boldsymbol{a}}_d(C, P)$ is one-dimensional, the stratification from §5.1 consists of only the stratum with $\boldsymbol{k}^+ = (1, \ldots, r+1)$. Hence the curve $W^{\boldsymbol{a}}_d(C, P)$ is necessarily smooth. Expanding formula (20), one has that $T_y\left(W^{\boldsymbol{a}}_d(C, P)\right)$ is

$$\left(1 + \frac{1}{2}\left((r+1)\theta + \sum_{k=1}^{r+1}(\lambda_k - r - 2 + k)T_k\right)(y-1)\right)\left|c_{\lambda_i+j-i}\right|_{1\leq i,j\leq r+1}. \tag{22}$$

One computes

$$\theta\left|c_{\lambda_i+j-i}\right|_{1\leq i,j\leq r+1} = g!\left|\frac{1}{(\lambda_i+j-i)!}\right|_{1\leq i,j\leq r+1},$$
$$T_k\left|c_{\lambda_i+j-i}\right|_{1\leq i,j\leq r+1} = g!\left|\frac{1}{(\lambda_i+\delta_{i,k}+j-i)!}\right|_{1\leq i,j\leq r+1}$$

where $\delta_{i,k}$ is the Kronecker delta: $\delta_{i,k} = 1$ for $i = k$, and $\delta_{i,k} = 0$ otherwise. The determinants can be computed via the following application of the Vandermonde identity:

$$\left|\frac{1}{(l_i+j-i)!}\right|_{1\leq i,j\leq r+1} = \frac{\prod_{1\leq i<j\leq r+1}(l_i - l_j + j - i)}{\prod_{i=1}^{r+1}(l_i + r + 1 - i)!}.$$

Using the combinatorial identity

$$(r+1)\left|\frac{1}{(\lambda_i+j-i)!}\right|_{1\leq i,j\leq r+1} = \sum_{k=1}^{r+1}(\lambda_k + r + 2 - k)\left|\frac{1}{(\lambda_i+\delta_{i,k}+j-i)!}\right|_{1\leq i,j\leq r+1},$$

the top degree part in (22) gives

$$\chi_y\left(W^{\boldsymbol{a}}_d(C, P)\right) = (y-1)\, g! \sum_{k=1}^{r+1}\lambda_k\left|\frac{1}{(\lambda_i+\delta_{i,k}+j-i)!}\right|_{1\leq i,j\leq r+1}.$$

For the curve $G^{\boldsymbol{a}}_d(C, P)$, the stratification in §6.1 consists of a single non-empty stratum, thus $\chi_y\left(G^{\boldsymbol{a}}_d(C, P)\right) = \chi_y\left(W^{\boldsymbol{a}}_d(C, P)\right)$. As a check, for $y = 0$ one recovers the *holomorphic* Euler characteristic computed in [6, §4.2].

7.5. **The surface case when $\lambda_1 = \cdots = \lambda_{r+1}$.** Let us consider the surface case in the *classical* Brill-Noether setting, that is, with no special ramification required at the marked point. Thus one has $a_i = i$ for $0 \leq i \leq r$, and $\lambda_1 = \cdots = \lambda_{r+1} = g - d + r =: \lambda$. In this case, one simplifies the notation as $W^r_d(C) := W^{\boldsymbol{a}}_d(C, P)$, and similarly, $G^r_d(C) :=$

$G_d^{\boldsymbol{a}}(C,P)$. The degree-two operators with non-zero action are:

$$\begin{aligned}
R_1^2\,|c_{\lambda+j-i}|_{1\le i,j\le r+1} &= \frac{(r+1)(r+2)}{2(\lambda+r+1)(\lambda+r+2)}\,g!\prod_{i=0}^{r}\frac{i!}{(\lambda+i)!},\\
R_1R_2\,|c_{\lambda+j-i}|_{1\le i,j\le r+1} &= \frac{r(r+1)}{2(\lambda+r)(\lambda+r+1)}\,g!\prod_{i=0}^{r}\frac{i!}{(\lambda+i)!},\\
R_2^2\,|c_{\lambda+j-i}|_{1\le i,j\le r+1} &= -R_1R_2\,|c_{\lambda+j-i}|_{1\le i,j\le t}\,, \qquad (23)\\
\theta\,R_1\,|c_{\lambda+j-i}|_{1\le i,j\le r+1} &= \frac{r+1}{\lambda+r+1}\,g!\prod_{i=0}^{r}\frac{i!}{(\lambda+i)!},\\
\theta^2\,|c_{\lambda+j-i}|_{1\le i,j\le r+1} &= g!\prod_{i=0}^{r}\frac{i!}{(\lambda+i)!}.
\end{aligned}$$

We are now ready to prove Corollary 4:

Proof of Corollary 4. The stratification from §5.1 consists of two strata: the full-dimensional stratum with $\boldsymbol{k}^+=\boldsymbol{k}=(1,2,\dots,r+1)$, and the codimension two stratum with $\boldsymbol{k}^+=(2,2,\dots,r+1)$ whose class is equal to $R_1R_2\,|c_{\lambda+j-i}|_{1\le i,j\le r+1}$. From Corollary 7.1, the motivic Hirzebruch class is given by

$$T_y\left(W_d^r(C)\right) = A_{\boldsymbol{\lambda}} - \chi_y\left(\mathbb{A}^1_{\mathbb{C}}\right)\,R_1R_2\,|c_{\lambda+j-i}|_{1\le i,j\le r+1}\,. \qquad (24)$$

As in the proof of Theorem 5.1, the affine space $\mathbb{A}^1_{\mathbb{C}}$ here coincides with the maximal Schubert cell inside the Schubert variety given by the generic fiber of $\widetilde{\Omega}_{\boldsymbol{\tau}}$ over the codimension-two stratum in $W_d^r(C)$.

One has $\chi_y\left(\mathbb{A}^n_{\mathbb{C}}\right)=(-y)^n$ (Example 2.5). On a surface, the power series $Q_y(\alpha)$ restricts as:

$$Q_y(\alpha) = 1+\frac{1}{2}\,\alpha(1-y)+\frac{1}{12}\,\alpha^2(1+y)^2.$$

The resulting expansion of (24) is

$$\begin{aligned}
T_y\left(W_d^r(C)\right) \;=\; &\Bigg(1+\frac{1}{2}\left((\lambda-r-1)R_1+(\lambda-r)R_2+(r+1)\theta\right)(y-1)\\
&+\frac{1}{24}\Bigg(\Big(\left(3(\lambda-r)^2+7r-5\lambda+2\right)(y-1)^2\\
&+8(1-r-\lambda)y\Big)R_1^2\\
&+2\left((3(\lambda-r)(\lambda-r-1)-1)(y-1)^2+20y\right)R_1R_2\\
&+\left(3(\lambda-r)^2(y-1)^2+(\lambda+r-2)\left((y-1)^2-8y\right)\right)R_2^2\\
&+2\left((3\lambda(r+1)-3r(r+2)-2)(y-1)^2-8y\right)\theta R_1\\
&+2\left((3\lambda(r+1)-3r(r+1)+1)(y-1)^2-8y\right)\theta R_2\\
&+3(r+1)^2(y-1)^2\theta^2\Bigg)\Bigg)\,|c_{\lambda+j-i}|_{1\le i,j\le r+1}\,.
\end{aligned}$$

Using (23), this gives the statement.

For the surface $G^r_d(C)$, as the stratification in §6.1 consists of a single non-empty stratum, Corollary 7.2 implies $\pi_* T_y(G^r_d(C)) = T_y(W^r_d(C))$, hence $\chi_y(G^r_d(C)) = \chi_y(W^r_d(C))$. □

Remark 7.3. We emphasize how the stratification from §5.1 for $W^r_d(C)$ is finer than the one considered in [34]. Indeed, for this example we have two strata, while the stratification used in [34] for the same locus consists of only one stratum, as $W^r_d(C)$ is a smooth surface.

7.6. **The surface case when $r = 1$.** Here we consider the surface case when $r = 1$. The case $\lambda_1 = \lambda_2$ being treated in §7.5, we assume here $\lambda_1 > \lambda_2$. The stratification from §5.1 consists of the single full-dimensional stratum with $\boldsymbol{k}^+ = (1, 2)$, contrary to the case $\lambda_1 = \lambda_2$. From Corollary 7.1, the motivic Hirzebruch class is given by

$$T_y\left(W^{\boldsymbol{a}}_d(C, P)\right) = A_{\boldsymbol{\lambda}}. \tag{25}$$

Formula (20) gives

$$\begin{aligned} T_y\left(W^{\boldsymbol{a}}_d(C, P)\right) \;=\; & \Bigg(1 + \frac{1}{2}\left((\lambda_1 - 2)R_1 + (\lambda_2 - 1)R_2 + 2\theta\right)(y-1) \\ & + \frac{1}{24}\Bigg(\left((3\lambda_1^2 - 11\lambda_1 + 12)(y-1)^2 - 8y\lambda_1\right) R_1^2 \\ & + 2\left((3(\lambda_1 - 2)\lambda_2 - 3\lambda_1 + 5)(y-1)^2 + 8y\right) R_1 R_2 \\ & + (\lambda_2 - 1)\left((3\lambda_2 - 2)(y-1)^2 - 8y\right) R_2^2 \\ & + 2\left((6\lambda_1 - 11)(y-1)^2 - 8y\right)\theta R_1 \\ & + 2\left((6\lambda_2 - 5)(y-1)^2 - 8y\right)\theta R_2 \\ & + 12(y-1)^2\theta^2\Bigg)\Bigg) \left|c_{\lambda_i + j - i}\right|_{1 \le i,j \le 2}. \end{aligned}$$

The degree-two operators are:

$$\begin{aligned} R_1^2 \left|c_{\lambda_i+j-i}\right|_{1\le i,j\le 2} &= g!\frac{3 + \lambda_1 - \lambda_2}{(\lambda_1+3)!\lambda_2!}, \\ R_1 R_2 \left|c_{\lambda_i+j-i}\right|_{1\le i,j\le 2} &= g!\frac{1 + \lambda_1 - \lambda_2}{(\lambda_1+2)!(\lambda_2+1)!}, \\ R_2^2 \left|c_{\lambda_i+j-i}\right|_{1\le i,j\le 2} &= g!\frac{\lambda_1 - \lambda_2 - 1}{(\lambda_1+1)!(\lambda_2+2)!}, \\ \theta\, R_1 \left|c_{\lambda_i+j-i}\right|_{1\le i,j\le 2} &= g!\frac{2 + \lambda_1 - \lambda_2}{(\lambda_1+2)!\lambda_2!}, \\ \theta\, R_2 \left|c_{\lambda_i+j-i}\right|_{1\le i,j\le 2} &= g!\frac{\lambda_1 - \lambda_2}{(\lambda_1+1)!(\lambda_2+1)!}, \\ \theta^2 \left|c_{\lambda_i+j-i}\right|_{1\le i,j\le 2} &= g!\frac{1 + \lambda_1 - \lambda_2}{(\lambda_1+1)!\lambda_2!}. \end{aligned}$$

Using these, the motivic Hirzebruch class is:

Corollary 7.4. *Fix $g \geq 2$ and $\boldsymbol{a} = (0 \leq a_0 < a_1 \leq d)$ with $\rho(g,1,d,\boldsymbol{a}) = 2$. For a general smooth pointed curve (C,P) of genus g, one has*

$$\begin{aligned} T_y\left(W_d^{\boldsymbol{a}}(C,P)\right) = \frac{1+\lambda_1-\lambda_2}{(\lambda_1+1)!\lambda_2!}\,\theta^{g-2}\Bigg(1 + \frac{\left(\lambda_1^2-(\lambda_2-2)\lambda_1+2\right)\lambda_2}{(\lambda_1+2)(\lambda_2+1)}(y-1)\theta \\ + \Big(\big((2\lambda_2(\lambda_2+2)+1)\lambda_1^3 - (\lambda_2-4)(2\lambda_2(\lambda_2+2)+1)\lambda_1^2 \\ + (\lambda_2(10-\lambda_2(6\lambda_2+5))+3)\lambda_1 - 3\lambda_2(\lambda_2+1)^2\big)(y-1)^2 \\ - (\lambda_1+2)(\lambda_2+1)\left(\lambda_1^2+4\lambda_1-\lambda_2^2-2\lambda_2+3\right)y\Big) \\ \frac{1}{(1+\lambda_1-\lambda_2)(\lambda_1+2)(\lambda_1+3)(\lambda_2+1)(\lambda_2+2)}\theta^2\Bigg). \end{aligned}$$

We deduce the Hirzebruch χ_y-genus:

Corollary 7.5. *Fix $g \geq 2$ and $\boldsymbol{a} = (0 \leq a_0 < a_1 \leq d)$ with $\rho(g,1,d,\boldsymbol{a}) = 2$. For a general smooth pointed curve (C,P) of genus g, one has*

$$\begin{aligned} \chi_y\left(W_d^{\boldsymbol{a}}(C,P)\right) = \Big(\big((2\lambda_2(\lambda_2+2)+1)\lambda_1^3 - (\lambda_2-4)(2\lambda_2(\lambda_2+2)+1)\lambda_1^2 \\ + (\lambda_2(10-\lambda_2(6\lambda_2+5))+3)\lambda_1 - 3\lambda_2(\lambda_2+1)^2\big)(y-1)^2 \\ - (\lambda_1+2)(\lambda_2+1)\left(\lambda_1^2+4\lambda_1-\lambda_2^2-2\lambda_2+3\right)y\Big)\frac{g!}{(\lambda_1+3)!(\lambda_2+2)!}. \end{aligned}$$

The stratification in §6.1 consists of a single non-empty stratum, hence $\chi_y(G_d^{\boldsymbol{a}}(C,P)) = \chi_y(W_d^{\boldsymbol{a}}(C,P))$.

Remark 7.6. Interestingly, the above formula for $\chi_y(W_d^{\boldsymbol{a}}(C,P))$ in the case $\lambda_1 > \lambda_2$ (i.e., $a_1 > a_0$) does not specialize to the formula for the case $\lambda_1 = \lambda_2$ from §7.5, unless $y = 0$.

The discrepancy arises from the fact that the stratification in the surface case with $r = 1$ consists of two strata when $\lambda_1 = \lambda_2$, while there is only one stratum when $\lambda_1 > \lambda_2$. Indeed, the stratum with $\boldsymbol{k}^+ = (2,2)$ has codimension two when $\lambda_1 = \lambda_2$, while it has codimension at least three when $\lambda_1 > \lambda_2$, hence it is empty on surfaces. Consequently, the motivic Hirzebruch class $T_y\left(W_d^{\boldsymbol{a}}(C,P)\right)$ in the surface case with $r = 1$ and $\lambda_1 > \lambda_2$ given by (24) does not specialize to the surface case with $\lambda_1 = \lambda_2$ given by (25), unless $y = 0$.

This is in contrast with the case $y = 0$, corresponding to the holomorphic Euler characteristic of a surface $W_d^{\boldsymbol{a}}(C,P)$, which following [6], does specialize from the case $\lambda_i > \lambda_{i+1}$ to the case $\lambda_i = \lambda_{i+1}$, for any i. When $r = 1$, this can be seen here since the second summand in (24) vanishes for $y = 0$.

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