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POSITIVITY OF EQUIVARIANT GROMOV-WITTEN INVARIANTS

DAVE ANDERSON AND LINDA CHEN

ABSTRACT. We show that the equivariant Gromov-Witten invariants of a projective homogeneous space $G/P$ exhibit Graham-positivity: when expressed as polynomials in the positive roots, they have nonnegative coefficients.

1. Introduction

Let $X = G/P$ be a projective homogeneous variety, for a complex reductive Lie group $G$ and parabolic subgroup $P$. Fix a maximal torus and Borel subgroup $T \subset B \subset P$, and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the corresponding set of simple roots, making the roots of $B$ positive. Let $W_P \subset W$ be the Weyl groups for $P$ and $G$, respectively. Let $B^-$ be the opposite Borel subgroup. The classes of the Schubert varieties $X(w) = BwP/P$ and opposite Schubert varieties $Y(w) = B^-wP/P$ give Poincaré dual bases of the equivariant cohomology ring $H^*_T X$, as $w$ ranges over the set $W_P$ of minimal coset representatives for $W/W_P$. Write $x(w) = [X(w)]^T$ and $y(w) = [Y(w)]^T$ for these classes.

A positivity property for multiplication in these bases was proved by Graham:

**Theorem 1.1 ([G]).** Writing

$$y(u) \cdot y(v) = \sum_w c^w_{u,v} y(w)$$

in $H^*_T X$, the coefficient $c^w_{u,v}$ lies in $\mathbb{N}[\alpha_1, \ldots, \alpha_n]$.

Following [K], the equivariant Gromov-Witten invariants are defined as follows. Let $d \in H_2(X, \mathbb{Z})$ be an effective class; taking the basis of Schubert curves $x(s_\alpha)$, one can identify $d$ with a tuple of nonnegative integers $(d_1, \ldots, d_k)$. Let $\overline{M} = \overline{M}_{0,r+1}(X, d)$ denote the Kontsevich moduli space of stable maps. This comes with $r + 1$ evaluation maps $ev_i : \overline{M} \to X$, as well as the standard map $\pi : \overline{M} \to \text{pt}$. 

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Definition 1.2. The equivariant Gromov-Witten invariant associated to classes $\sigma_1, \ldots, \sigma_{r+1}$ is
\[
I_T^d(\sigma_1 \cdots \sigma_{r+1}) := \pi_T^* (ev_1^* \sigma_1 \cdots ev_{r+1}^* \sigma_{r+1})
\]
in $H^*_T(pt)$, where $\pi_T^*$ is the equivariant pushforward $H^*_T(M) \to H^*_T(pt)$.

When $r = 2$, these define equivariant quantum Littlewood-Richardson (EQLR) coefficients:
\[
c_{w,d}^{u,v} = I_T^d(y(u) \cdot y(v) \cdot x(w)).
\]
The EQLR coefficients were shown to be Graham-positive, in the sense of Theorem 1.1, by Mihalcea in [M]. Remarkably, they define an associative product in the equivariant (small) quantum cohomology ring $QH^*_T(X)$, via
\[
y(u) \circ y(v) = \sum_{w,d} q^d c_{u,v}^{w,d} y(w),
\]
so Mihalcea’s result is a generalization of Graham’s to the setting of equivariant quantum Schubert calculus.

In this note, we will show that the multiple-point equivariant Gromov-Witten invariants are Graham-positive:

**Theorem 1.3.** For any elements $v_1, \ldots, v_r, w \in W^P$, the equivariant Gromov-Witten invariant
\[
I_T^d(y(v_1) \cdots y(v_r) \cdot x(w))
\]
lies in $\mathbb{N}[\alpha_1, \ldots, \alpha_n]$.

Associativity of the equivariant quantum ring $QH^*_T(X)$ defines (generalized) EQLR coefficients $c_{v_1,\ldots,v_r}^{w,d}$:
\[
y(v_1) \circ \cdots \circ y(v_r) = \sum_{w,d} q^d c_{v_1,\ldots,v_r}^{w,d} y(w).
\]

By induction using the $r = 2$ case of Theorem 1.3, it follows that these EQLR coefficients are also Graham-positive; indeed, the associativity relations are subtraction-free. This gives a new proof of Mihalcea’s positivity theorem. For $r > 2$, however, the EQLR coefficients $c_{v_1,\ldots,v_r}^{w,d}$ are not the same as the equivariant Gromov-Witten invariants in Theorem 1.3.

The proof of Theorem 1.3 is given in §4; the idea is to represent the coefficients of this polynomial as degrees of effective zero-cycles, using a transversality argument (Theorem 4.4). An inspection of Mihalcea’s proof of positivity for EQLR coefficients suggests that his method should also work for Gromov-Witten invariants, but we find our geometric interpretation of the coefficients appealing. Moreover, we use the dimension estimates from §4 to derive a Giambelli formula for $QH^*_T(SL_n/P)$ in [AC].

**Remark 1.4.** As in [G], there is a corresponding positivity theorem with the roles of positive and negative roots interchanged: the Gromov-Witten invariants $I_d^T(x(v_1) \cdots x(v_r) \cdot y(w))$ lie in $\mathbb{N}[-\alpha_1, \ldots, -\alpha_n]$. All the arguments
proceed in exactly the same manner. In fact, it is this version (for \( r = 2 \)) that is treated in [M].

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2. Setup

We assume \( G \) is an adjoint group, so that the simple roots \( \Delta = \{ \alpha_1, \ldots, \alpha_n \} \) form a basis for the character group of \( T \). We fix the basis \( -\Delta = \{ -\alpha_1, \ldots, -\alpha_n \} \) of negative simple roots, and use it to identify \( T \) with \( \mathbb{C}^n \).

2.1. Equivariant cohomology. Let \( ET \to BT \) be the universal principal \( T \)-bundle; that is, \( ET \) is a contractible space with a free right \( T \)-action, and \( BT = ET/T \). By definition, the equivariant cohomology of a \( T \)-variety \( Z \) is the ordinary (singular) cohomology of the Borel mixing space \( ET \times^T Z \). (This notation means quotient by the relation \((e \cdot t, z) \sim (e, t \cdot z)\).) While \( ET \) is infinite-dimensional, it may be approximated by finite-dimensional smooth varieties. We will set \( E = (\mathbb{C}^m \setminus \{0\})^n \), with \( T = (\mathbb{C}^*)^n \) acting by scaling each factor. For fixed \( k \) and \( m \gg 0 \), one has natural isomorphisms

\[
H^*_T Z := H^*(ET \times^T Z) \cong H^*(E \times^T Z),
\]

so any given computation may be done with these approximation spaces.

Note that \( B = E/T \) is isomorphic to \((\mathbb{P}^{m-1})^n\). For a \( T \)-variety \( Z \), we will generally use calligraphic letters to denote the corresponding approximation space: \( Z = E \times^T Z \), always understanding a suitably large fixed \( m \). This is a fiber bundle over \( B \), with fiber \( Z \).

For each \( j = 0, \ldots, m - 1 \), we fix transverse linear subspaces \( \mathbb{P}^{m-1-j} \) and \( \mathbb{P}^j \) inside \( \mathbb{P}^{m-1} \), and for each multi-index \( J = (j_1, \ldots, j_n) \) with \( 0 \leq j_i \leq m - 1 \), we set

\[
B_J = \mathbb{P}^{j_1} \times \cdots \times \mathbb{P}^{j_n} \quad \text{and} \quad B^J = \mathbb{P}^{m-1-j_1} \times \cdots \times \mathbb{P}^{m-1-j_n}.
\]

So \( \dim B_J = \text{codim} B^J = |J| := j_1 + \cdots + j_n \). Similarly, write \( Z_J = (\pi^T)^{-1} B_J \) and \( Z^J = (\pi^T)^{-1} B^J \), where \( \pi^T : Z \to B \) is the projection. The notation is chosen to suggest an identification of the pushforward for this fiber bundle with the equivariant pushforward \( \pi_*^T : H^*_T Z \to H^*_B(\text{pt}) \).

Let \( O_i(-1) \) be the tautological bundle on the \( i \)th factor of \( B = (\mathbb{P}^{m-1})^n \). The choice of basis \(-\Delta\) for the character group of \( T \) yields an equality \( \alpha_i = c_1(O_i(1)) \). If \( \alpha = a_1 \alpha_1 + \cdots + a_n \alpha_n \) is a root, we will sometimes write \( O(\alpha) = O_i(a_1) \otimes \cdots \otimes O_n(a_n) \) for the corresponding line bundle, so \( c_1(O(\alpha)) = \alpha \). Note that \( O(\alpha) \) is globally generated if and only if \( \alpha \) is a positive root.

From the definitions, we have

\[
[B^J] = \alpha^J := \alpha_1^{j_1} \cdots \alpha_n^{j_n}
\]
in $H^*\mathbb{B}$. As a consequence, suppose $c = \sum c_J \alpha^J$ is an element of $H^*\mathbb{B} = H^*_T(pt)$, with $c_J \in \mathbb{Z}$. Using Poincaré duality on $\mathbb{B}$, we have $c_J = \pi^\mathbb{B}_*(c \cdot [B_J])$, where $\pi^\mathbb{B}$ is the map $\mathbb{B} \to pt$.

When $c = \pi^T_*(\sigma)$ comes from a class $\sigma \in H^*_T \mathbb{Z} = H^*_T \mathbb{Z}$ for a complete $T$-variety $Z$, we have

\[ c_J = \pi_Z^*(\sigma \cdot [Z_J]), \]

using the projection formula and the fact that $(\pi^T)^*[\mathbb{B}_J] = [Z_J]$. (The latter holds since $\pi^T: Z \to \mathbb{B}$ is flat; for a more general argument in the case where $Z$ is Cohen-Macaulay, see [FPr, Lemma, p. 108].)

2.2. Stable maps. We briefly summarize some basic facts about the space of stable maps; proofs and details may be found in [FPa]. As always, $X = G/P$. The (coarse) moduli space $\mathcal{M} = \mathcal{M}_{0,r+1}(X,d)$ parametrizes data $(f, C, p_1, \ldots, p_{r+1})$, where $C$ is a connected nodal curve of genus 0, and $f: C \to X$ is a map with $f_*[C] = d$ in $H_2(X,\mathbb{Z})$. (Stability means that any irreducible component of $C$ which is collapsed by $f$ has at least three “special” points, i.e., marked points $p_i$ or nodes.)

The space of stable maps is an irreducible projective variety of dimension

\[ \dim \mathcal{M} = \dim X + \langle c_1(TX), d \rangle + r - 2, \]

and has quotient singularities, and therefore rational singularities; in particular, it is Cohen-Macaulay. The locus parametrizing maps with irreducible domain is a dense open subset $M = M_{0,r+1}(X,d) \subseteq \mathcal{M}$, and the complement is a divisor $\partial \mathcal{M} = \mathcal{M} \setminus M$.

There are natural evaluation maps $ev_i : \mathcal{M} \to X$, defined by sending a stable map $(f, C, p_1, \ldots, p_{r+1})$ to $f(p_i)$. The group $G$ acts on $\mathcal{M}$ by $g \cdot (f, C, \{p_i\}) = (g \cdot f, C, \{p_i\})$, and the evaluation maps are equivariant for the actions of $G$ on $\mathcal{M}$ and $X$. Considering the induced action of $T < G$, we obtain maps $ev^T_i : \mathcal{M} \to X$ on Borel mixing spaces, which commute with the projections to $\mathbb{B}$.

**Remark 2.1.** The significance of $\mathcal{M}$ being Cohen-Macaulay is that the usual apparatus of intersection theory applies; see especially Lemma 4.2 below. In fact, the corresponding moduli stack is smooth, so one could argue directly using intersection theory on stacks.

3. A GROUP ACTION

In [A] and [AGM], a large group action on the mixing space $X$ was constructed; we describe it here. The idea is to mix the transitive action of $(PGL_m)^n$ on $\mathbb{B}$ with a “fiberwise” action by Borel groups. Let $T$ act on $G$ by conjugation, and let $\mathcal{G} = E \times^T G$ be the corresponding group scheme over $\mathbb{B}$. Because $T$ acts by conjugation, the evident action $(E \times G) \times_{E \times E} (E \times X) \to E \times X$ descends to an action $\mathcal{G} \times_\mathbb{B} X \to X$.

Let $U \subset B \subset G$ be the unipotent radical of $B$, and let $U \subset B \subset \mathcal{G}$ be the corresponding group bundles over $\mathbb{B}$. As a variety, $U$ is isomorphic to the
vector bundle $\bigoplus_{\alpha \in \mathbb{R}^+} \mathcal{O}(\alpha)$ on $\mathbb{B}$, where the sum is over the positive roots. Now consider the group of sections $\Gamma_0 = \text{Hom}_{\mathbb{B}}(\mathcal{B}, \mathcal{U})$; this is a connected algebraic group over $\mathbb{C}$. As observed in §2.1, each $\mathcal{O}(\alpha)$ is globally generated. It follows that for each $x \in \mathbb{B}$, the map $\Gamma_0 \to \mathcal{U}_x$ given by evaluating sections at $x$ is surjective, and therefore we have:

**Lemma 3.1** ([AGM, Lemma 6.3]). Let $\Gamma$ be the mixing group $\Gamma_0 \rtimes (\text{PGL}_m)^n$, where $(\text{PGL}_m)^n$ acts on $\mathcal{B}$ via its action on $\mathbb{B}$. Then $\Gamma$ is a connected linear algebraic group acting on $\mathcal{X}$, with (finitely many) orbits whose closures are the Schubert bundles $\mathcal{X}(w)$.

Similarly, the group $\Gamma^{(r)} = \Gamma_0^r \rtimes (\text{PGL}_m)^n$ acts on the $r$-fold fiber product $\mathcal{X} \times_{\mathbb{B}} \cdots \times_{\mathbb{B}} \mathcal{X}$, with orbit closures $\mathcal{X}(w_1) \times_{\mathbb{B}} \cdots \times_{\mathbb{B}} \mathcal{X}(w_r)$.

### 4. Transversality

A map $f : Y \to X$ is said to be **dimensionally transverse** to a subvariety $W \subseteq X$ if $\text{codim}_Y(f^{-1}W) = \text{codim}_X(W)$. We will need the following version of Kleiman’s transversality theorem; see [Kl] and [S]. As a matter of notation, if a group $\Gamma$ acts on $X$, we write $\gamma f : \gamma Y \to X$ for the composition $Y \to X \to Y$, i.e., the translation of $f$ by the action of $\gamma \in \Gamma$.

**Proposition 4.1.** Let $\Gamma$ be a group acting on a smooth variety $X$, and suppose $f : Y \to X$ is dimensionally transverse to the orbits of $\Gamma$. Assume $Y$ is Cohen-Macaulay. Let $g : Z \to X$ be any map. Then for a general element $\gamma \in \Gamma$, the fiber product $V_\gamma = \gamma Y \times_X Z$ has dimension equal to $\dim Y + \dim Z - \dim X$.

The essential point in the proof is that the hypotheses imply the map $\Gamma \times Y \to X$ is flat.

We will also use the following lemma:

**Lemma 4.2** ([FPr, Lemma, p. 108]). Let $f : Z \to X$ be a morphism from a pure-dimensional Cohen-Macaulay scheme $Z$ to a nonsingular variety $X$, and let $W \subseteq X$ be a closed Cohen-Macaulay subscheme of pure codimension $d$. Let $V = f^{-1}W$, and assume $\text{codim}_Z(V) = d$. Then $V$ is Cohen-Macaulay, and $f^*[W] = [V]$.

Now resume the previous notation, so $X = G/P$ and $\overline{M} = \overline{M}_{0,r+1}(X, \mathbf{d})$. Since each evaluation map $\text{ev}_i : \overline{M} \to X$ is $G$-equivariant, it is flat. If $W \subseteq X$ is any Cohen-Macaulay subscheme of codimension $d$, it follows that $\text{ev}_i^{-1}W \subseteq \overline{M}$ has the same properties, and similarly, $(\text{ev}_i^T)^{-1}W \subseteq \overline{M}$. In particular, the subscheme

$$Z = (\text{ev}_{r+1}^T)^{-1}(\mathcal{X}(w)) \subseteq \overline{M}$$

is Cohen-Macaulay of codimension $\dim X - \ell(w)$, and we have $[Z] = (\text{ev}_{r+1}^T)^*(x(w))$ by Lemma 4.2. Similarly, we have

$$(1) \quad [Z] = (\text{ev}_{r+1}^T)^*(x(w)) \cdot [\overline{M}]$$
Consider the map $\text{ev} = \text{ev}_1 \times \cdots \times \text{ev}_r : \overline{M} \to X^r$ and the corresponding map on mixing spaces $\text{ev}^T : \overline{M} \to X^r$. Let $Y = \gamma(v_1) \times_B \cdots \times_B \gamma(v_r)$, and let $f$ be the inclusion of $Y$ in the $r$-fold fiber product $X^r$.

**Lemma 4.3.** Let $\gamma = (\gamma_1, \ldots, \gamma_r)$ be a general element in $\Gamma^{(r)}$.

(a) The intersection

$$V_\gamma = (\text{ev}_1^T)^{-1}(\gamma_1 \gamma(v_1)) \cap \cdots \cap (\text{ev}_r^T)^{-1}(\gamma_r \gamma(v_r)) \cap Z_J$$

is Cohen-Macaulay and pure-dimensional, of dimension $\dim \overline{M} + |J| - \dim X + \ell(w) - \ell(v_1) - \cdots - \ell(v_r)$. (In the fiber product, $Z_J$ maps to $X^r$ by the restriction of $\text{ev}^T$.)

(b) Similarly, the intersection

$$\partial V_\gamma = (\text{ev}_1^T)^{-1}(\gamma_1 \gamma(v_1)) \cap \cdots \cap (\text{ev}_r^T)^{-1}(\gamma_r \gamma(v_r)) \cap Z_J \cap \partial \overline{M}$$

has pure dimension $\dim \overline{M} + |J| - \dim X + \ell(w) - \ell(v_1) - \cdots - \ell(v_r) - 1$.

In particular, when $\dim \overline{M} + |J| - \dim X + \ell(w) - \ell(v_1) - \cdots - \ell(v_r) = 0$, the intersection $V_\gamma$ consists of finitely many points contained in $\overline{M}$.

**Proof.** Note that $Z_J$ is Cohen-Macaulay (since $Z$ is), of dimension $\dim \overline{M} + |J| - \dim X + \ell(w)$. Each opposite Schubert bundle $\gamma(v)$ intersects each $\Gamma$-orbit closure $X(w)$ properly, so the map $f : Y \hookrightarrow X^r$ is dimensionally transverse to the $\Gamma^{(r)}$-orbits. The first statement follows by an application of Proposition 4.1.

The second statement is proved similarly; note that the divisor $\partial \overline{M}$ is Cohen-Macaulay and $G$-invariant, and the same argument as before shows that $Z_J \cap \partial \overline{M}$ is a Cohen-Macaulay divisor in $Z_J$. \hfill $\square$

We can now prove Theorem 1.3. In fact, it follows immediately from (a), together with a more precise statement.

**Theorem 4.4.** Write $I^*_A(y(v_1) \cdots y(v_r) \cdot x(w)) = \sum c_J \alpha^J$ in $H^*_\text{pt}(\overline{M})$. Then, with notation as in Lemma 4.3, we have

$$c_J = \deg(V_\gamma)$$

when $\dim \overline{M} + |J| - \dim X + \ell(w) - \ell(v_1) - \cdots - \ell(v_r) = 0$, and $c_J = 0$ otherwise.

In particular, since $V_\gamma$ is an effective cycle, $c_J$ is a nonnegative integer.

**Proof.** Using (a) from §2.1, we have

$$c_J = \pi_{*\overline{M}}((\text{ev}_1^T)^*y(v_1) \cdots (\text{ev}_r^T)^*y(v_r) \cdot (\text{ev}_{r+1}^T)^*x(w) \cdot [\overline{M}])$$

The claim is that $(\text{ev}_1^T)^*y(v_1) \cdots (\text{ev}_r^T)^*y(v_r) \cdot (\text{ev}_{r+1}^T)^*x(w) \cdot [\overline{M}] = [V_\gamma]$ in $H^*\overline{M}$. 

First observe that \((ev^T_r)^* y(v_1) \cdots (ev^T_r)^* y(v_r) = (ev^T)^* y(v_1)\times \cdots \times y(v_r)\). Since \(\Gamma^{(r)}\) is connected, we have 
\[
[\gamma \mathcal{Y}] = [\mathcal{Y}] = y(v_1) \times \cdots \times y(r) \in H^*(X^r) = H^*_T(X^r).
\]
By the same argument as in the paragraph after Lemma 4.2, we have 
\[
[(ev^T_r)^{-1} (\gamma \mathcal{Y})] = (ev^T)^* y(v_1) \times \cdots \times y(v_r)).
\]
By (1), we have \([Z_J] = (ev^T_{r+1})^* x(w) \cdot [\mathcal{M}]_J\). Since \((ev^T)^{-1} (\gamma \mathcal{Y})\) and \(Z_J\) intersect properly in \(V_\gamma\) by Lemma 4.3, we have \([(ev^T)^{-1} (\gamma \mathcal{Y})] \cdot [Z_J] = [V_\gamma]\), as desired.

**Remark 4.5.** Let \(\mathcal{M}_{0,r+1}\) be the moduli space of stable curves with \(r + 1\) marked points; this is a nonsingular projective variety of dimension \(r - 2\). Since \(T\) acts trivially on this space, the corresponding mixing space is \(\mathcal{M}_{0,r+1} = \mathbb{B} \times \mathcal{M}_{0,r+1}\). The forgetful map \(\varphi : \mathcal{M} \to \overline{\mathcal{M}}_{0,r+1}\) induces a map \(\overline{\mathcal{M}} \to \overline{\mathcal{M}}_{0,r+1}\). Let \(\bar{\varphi} : \overline{\mathcal{M}} \to \overline{\mathcal{M}}_{0,r+1}\) be the composition with the second projection, and for \(x \in \overline{\mathcal{M}}_{0,r+1}\), write \(\mathcal{M}(x) = \bar{\varphi}^{-1}(x)\). Using the notation of Lemma 4.3, the same arguments used in the proof of the lemma also establish the following dimension counts:

(a) Let \(V_\gamma(x) = V_\gamma \cap \mathcal{M}(x)\). Then \(V_\gamma(x)\) is Cohen-Macaulay, of pure dimension \(\dim \overline{\mathcal{M}} + |J| - (\dim X - \ell(w)) - \ell(v_1) - \cdots - \ell(v_r) - (r - 2)\).

(b) Let \(\partial V_\gamma(x) = \partial V_\gamma \cap \mathcal{M}(x)\). Then \(\partial V_\gamma(x)\) is Cohen-Macaulay, of pure dimension \(\dim \overline{\mathcal{M}} + |J| - (\dim X - \ell(w)) - \ell(v_1) - \cdots - \ell(v_r) - (r - 2) - 1\).

**References**


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