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Some Unusual Locus Problems

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Most locus problems one sees are distance problems, and in geometry texts distance almost always means either distance between points or distance between a point and a line. These two distance concepts are special cases of a more general concept, the distance between two point sets. Once one understands this more general concept, many new distance locus problems arise. The solutions are often surprising (at least they were to me) and they provide a good review of the locus properties of the conic sections.

In this article, we explain this more general distance concept and give several examples of its use, leaving more and more of the justification to the reader as we go along.

To give an idea where we are headed, here is the last problem we will analyze: Find the "locus" of all circles equidistant from two given circles.

We will assume that all point sets are on the plane and that the distance between points $\mathbf{a} = (x, y)$ and $\mathbf{b} = (x', y')$ has been defined as usual:

$$d(\mathbf{a}, \mathbf{b}) = \sqrt{(x - x')^2 + (y - y')^2}$$

If A and B are point sets, define their distance $d(A, B)$ to be

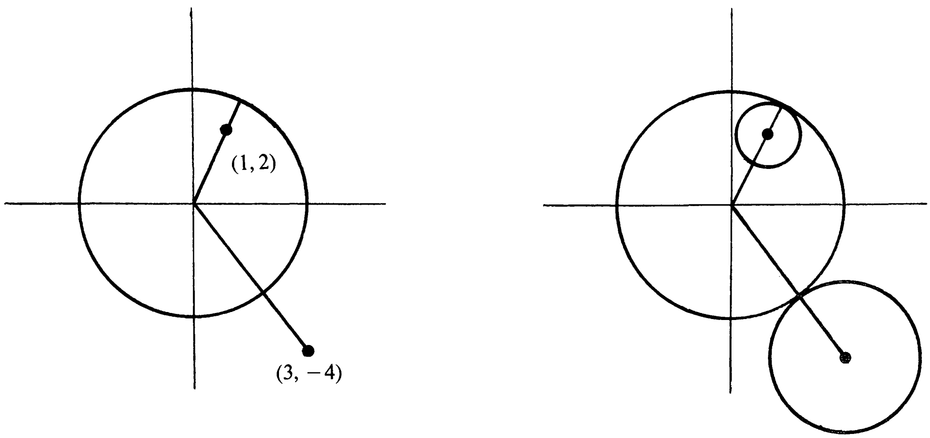
$$\text{glb}\{\, d(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} \in A, \mathbf{b} \in B \,\},$$

where as usual glb means greatest lower bound. Also, we define the distance $d(\mathbf{a}, B)$ from point $\mathbf{a}$ to set B to be $d(\{\mathbf{a}\}, B)$.

The glb need not be *in* the set of point distances; that is, there need not exist a closest pair $\mathbf{a} \in A$ and $\mathbf{b} \in B$ with $d(\mathbf{a}, \mathbf{b}) = d(A, B)$. For instance, let A be the line

$y = \pi/2$ and B be the graph of $y = \text{Arctan}\, x$. Then $d(A, B) = 0$ although $A \cap B = \emptyset$. Another example: Let A be the open left half-plane $\{(x, y) \mid x < 0\}$ and B be the open right half-plane $\{(x, y) \mid x > 1\}$. Then $d(A, B) = 1$ yet $d(\mathbf{a}, \mathbf{b}) > 1$ for all $\mathbf{a} \in A$ and $\mathbf{b} \in B$. If these examples seem surprising, it is because when we think of point sets we generally think of closed bounded figures, and in that case one can show (in a course in analysis or general topology) that there *is* always a closest pair of points. (In fact, it suffices that only one set be bounded so long as both are closed. Also, if both sets are convex, one of them strictly, then the pair of closest points is unique.)

If $A = \{\mathbf{a}\}$ and $B = \{\mathbf{b}\}$, then clearly $d(A, B) = d(\mathbf{a}, \mathbf{b})$. If $A = \{\mathbf{a}\}$ and B is a straight line, then there is a closest point in B to $\mathbf{a}$, namely the foot of the perpendicular to B from $\mathbf{a}$; so by our definition, $d(A, B)$ is the length of this perpendicular. But this length is also precisely the special definition usually given in texts for the distance from a point to a line. Thus our general definition does include the two special cases mentioned earlier.



(A) (B)

Figure 1. Distance from a point to a circle.

To develop a feel for the general definition, we begin by letting C be (the point set of) the circle $x^2 + y^2 = 9$. What is the distance from $\mathbf{a} = (1, 2)$ to C and from $\mathbf{b} = (3, -4)$ to C? Draw radii of C through these points; see Figure 1(A). This suggests that $d(\mathbf{a}, C) = 3 - \sqrt{5}$ and $d(\mathbf{b}, C) = 5 - 3 = 2$. The smaller circles drawn around $\mathbf{a}$ and $\mathbf{b}$ in Figure 1(B) make clear that we do have the right answers, because no other points of C are as close to $\mathbf{a}$ and $\mathbf{b}$ respectively as the points of tangency of these two new circles to C.

It is equally easy to find the distance between a point and, say, a square or a triangle. However, as soon as the point set C becomes slightly more complicated, the problem gets much harder. For instance, what is the distance between (0, 1) and the parabola $y = x^2$? This is hard to answer without calculus. One can do it without calculus if one is given that the slope of $y = x^2$ at (x, x^2) is $2x$, and that the segment from a point $\mathbf{a}$ to its closest point $\mathbf{b}$ on a smooth curve without endpoints (e.g., C) is perpendicular to the curve at $\mathbf{b}$. The answer in this case is $\sqrt{3}/2$ and there are two closest points on the parabola, $(\pm\sqrt{1/2}, 1/2)$.

What is the distance between two circles if (1) they are external to each other, or (2) one is inside the other, or (3) they intersect? Hint: draw some pictures and connect the centers by line segments.

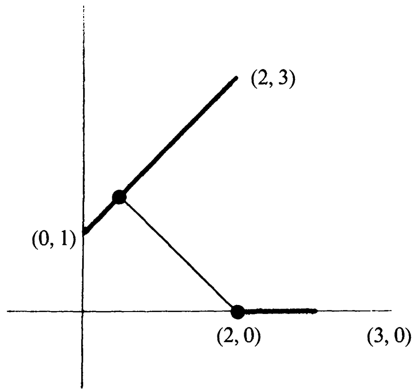


Figure 2. Distance between two line segments.

Now let us take lines and line segments for our sets. Clearly, the distance between two lines is 0 if they intersect and equals the length of any perpendicular between them if they are parallel. What is the distance between two line segments? That depends! In Figure 2, the distance is $\frac{3}{2}\sqrt{2}$ and the closest points are given as large dots. Can you see that, so long as the two segments are not parallel and don't intersect, at least one of the unique pair of closest points is an endpoint of its segment?

Finally, what is the distance between the parabolas $y = x^2 + 1$ and $x = y^2 + 1$? One can answer this using algebra and symmetry if one again knows that the slope at (x, y) on the first parabola is $2x$. Note: the parabolas are symmetric to each other through the line $y = x$.

Now let's do some locus problems. The simplest sort is: describe the locus of all points distance d from some set A. If A is a point **a**, we have a standard easy problem; the answer is the circle of radius d with center **a**. But what if A is itself a circle of radius r and center **a**? The answer depends on whether or not $d < r$. If it is, then the locus has two components, a circle of radius $r + d$ and a circle of radius $r - d$, both with center **a**. If $r < d$ (or if A is the *disk* of radius r and center **a**, i.e., A is all points on or inside the circle), then the locus of points distance d from A is just the circle around **a** of radius $r + d$.

What is the locus of points distance d from a square or a triangle? The latter problem is illustrated in Figure 3. The original triangle, drawn thicker, is equilateral of side s, with $d < s\sqrt{3}/6$. By picking various polygonal sets for A, one can get some very pretty pictures as loci—and some good area problems. For instance, what is the area of the region between the two components of the locus in Figure 3?

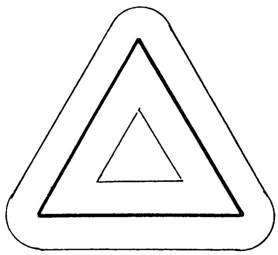


Figure 3. Points a fixed distance from a triangle.

Again, once one allows A to be curved, these questions become much more difficult. For instance, what is the locus of all points distance 1 from the parabola $y = x^2$? Here too the locus has two components, below the parabola and above it.

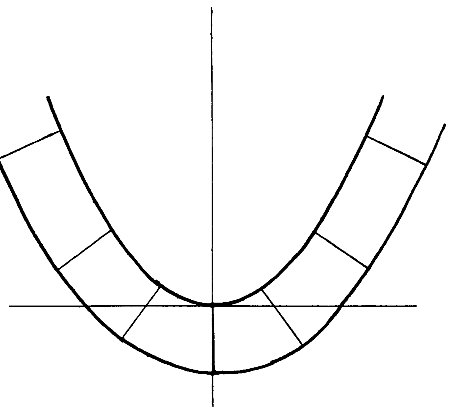


Figure 4. Construction of all points a fixed distance below a parabola.

The component below is easily described in words: for every point on the parabola draw an outward facing perpendicular of length 1; the points on the far ends of these segments form the component. See Figure 4. However, the same construction with inward facing perpendiculars gives too many points. For instance, it gives $(0, 1)$ at the end of the perpendicular from $(0, 0)$, and we have already remarked that $(0, 1)$ is not distance 1 from this parabola. The asymmetry between the inside and outside components of the locus is due to the fact that the parabola is concave up, not down. Question: Is the component of the locus below the parabola another parabola? A much harder question (but still doable by elementary means given the information about slopes on this parabola stated earlier): What points above the parabola *are* on the locus?

Now let us consider the locus of points equidistant from two sets A and B. The locus is well known if A and B are both points (it's the perpendicular bisector of the connecting segment) and if A is a point and B a line (a parabola). But there are many more possibilities to consider. Here are some interesting ones.

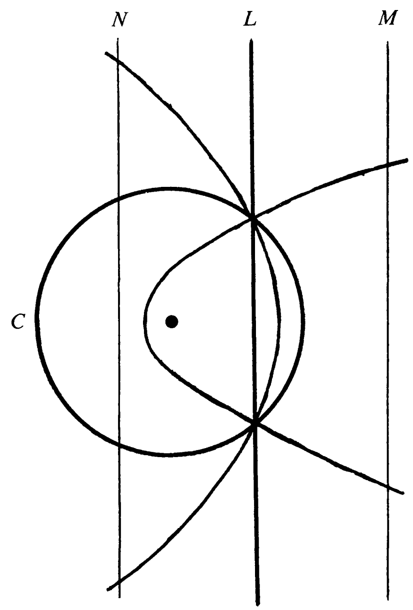
1. Two intersecting lines
2. Two rays with the same origin (be careful!)
3. Two line segments
4. Two mutually external circles
   a. of equal radius
   b. of different radii
5. A circle and a line outside it
6. A circle and an intersecting line
7. Two intersecting circles
8. A square and a straight line outside it

To give an indication of how to do these, let us sketch solutions to 6 and to the special case of 8 when the line is parallel to a side of the square.

The answers are shown in Figures 5 and 6. In Figure 5, the circle C has radius r and center **a**, and L is the original intersecting line. M and N are parallel lines distance r on either side of L. The locus is the union of two parabolas, one with focus **a** and directrix M, the other with focus **a** and directrix N. One way to show this is by considering four cases for where a locus point might be: either left or right of L and either inside or outside C. The union of the points from the four cases, along with the two points of intersection of L and C, gives the complete locus. We illustrate two of the cases. If **p** is outside C and right of L and equidistant from both, then **p** is equidistant from the point **a** and the line N, hence on the second parabola mentioned above. If **p** is inside C and right of L, then **p** is distance d from both sets iff it is distance $r - d$ from **a** and $r - d$ from M. Hence this case gives those points on the first parabola inside C. And so on.

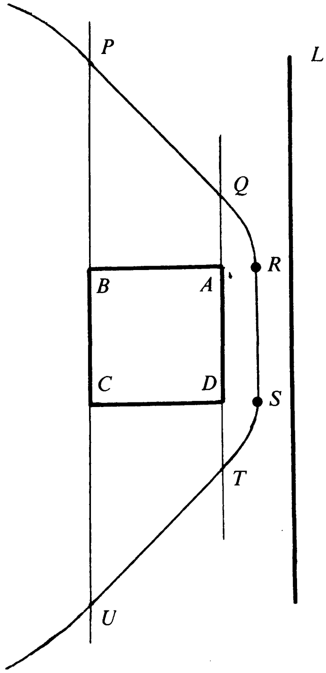
For problem 8, the curve $PQRSTU$ in Figure 6 is the locus. Segment RS is a straight line segment, halfway between AD and L. Segment QR is part of a parabola with focus A and directrix L. Segment PQ is linear; it is halfway between side BA of the square and L. The infinite section of the locus left of P is part of a parabola with focus B and directrix L. The lower segments of the locus are described similarly. Why is it that no point on the locus has its closest point of the square inside side BC?

Finally, although we have generalized the sets from which we find distances, we have still been looking for loci of *points* with distance properties relative to these sets. We could just as well ask for loci of *figures*. That is, we could pose problems like: Find all circles distance d from a given circle C of radius r. Or, find all circles equidistant from two given circles. The solution to the first problem depends on the relative sizes of d and r. We state the solution when $r > d$; there are more circles in the locus in this case. Let K and K' be circles concentric with C of radius $r - d$ and $r + d$ respectively. The circles distance d from C are: K, all circles internally tangent to K; K', all circles with K' internally tangent to them; and all circles externally tangent to K'. See Figure 7, where C is drawn heavy, K and K' are dotted, and the other circles are drawn normally. Why is this the (entire) solution?



Points equidistant from line L and circle C.

Figure 5.



Points equidistant from a line and a square.

Figure 6.

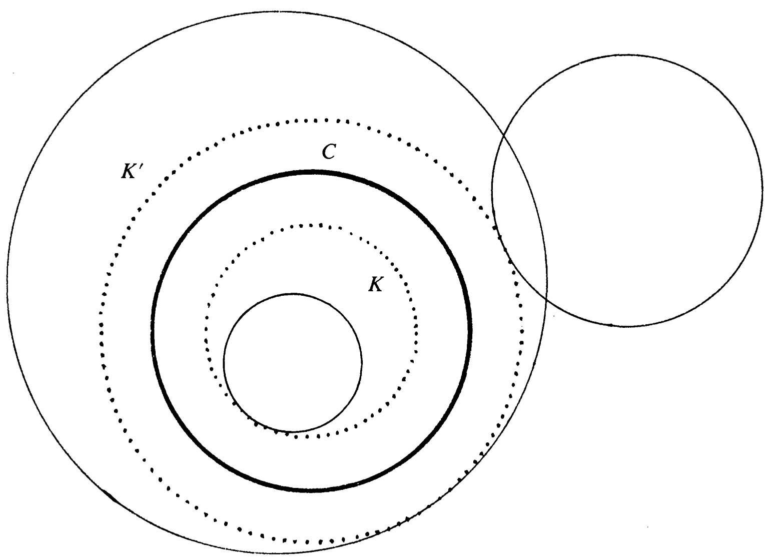


Figure 7. Circles a fixed distance from circle C.

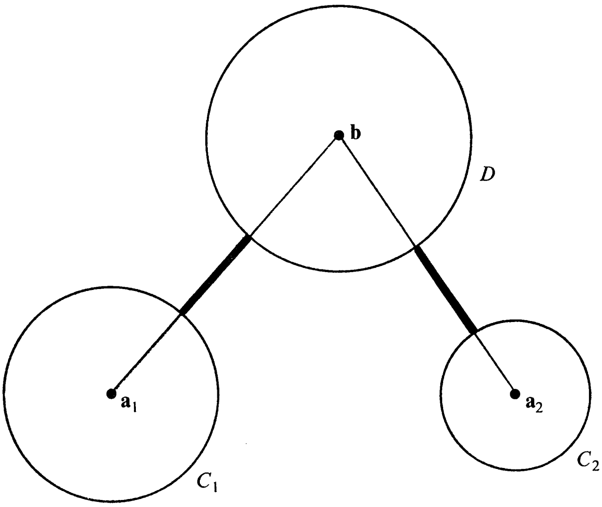


Figure 8. A circle externally equidistant from C_1 and C_2.

As for finding all circles equidistant from two given circles, let us take the case where the circles C_1, C_2 are mutually external, with radii $r_1 > r_2$, and with $d(C_1, C_2) > r_1$. Let $\mathbf{a}_1, \mathbf{a}_2$ be the two centers. The circles we seek, equidistant from these two, are of several types.

(1) Distance 0 from both C_1 and C_2, i.e., any circle intersecting both.

(2) Circles D inside C_1. There are none such equidistant from C_1 and C_2, since for any such D, $d(C_1, D) < r_1$ and $d(C_2, D) > d(C_2, C_1) > r_1$.

(3) Circles inside C_2. Again there are none.

(4) Circles D external to both C_1 and C_2. Let $\mathbf{b}$ be the center of such a circle and let R be its radius. As Figure 8 shows,

$$d(\mathbf{b}, \mathbf{a}_1) - R - r_1 = d(C_1, D) = d(C_2, D) = d(\mathbf{b}, \mathbf{a}_2) - R - r_2.$$

Thus

$$d(\mathbf{b}, \mathbf{a}_1) - d(\mathbf{b}, \mathbf{a}_2) = r_1 - r_2.$$

That is, the locus of centers $\mathbf{b}$ of such circles D is one branch of a hyperbola with foci $\mathbf{a}_1$ and $\mathbf{a}_2$! R is any value which keeps D external to both C_1 and C_2.

(5) Circles D containing C_1 and external to C_2. With $\mathbf{a}_1, \mathbf{a}_2, r_1, r_2, \mathbf{b}, R$ as before, we see from Figure 9 that

$$R - d(\mathbf{b}, \mathbf{a}_1) - r_1 = d(D, C_1) = d(D, C_2) = d(\mathbf{b}, \mathbf{a}_2) - R - r_2.$$

So

$$d(\mathbf{b}, \mathbf{a}_1) + d(\mathbf{b}, \mathbf{a}_2) = 2R + r_2 - r_1 > R + r_2.$$

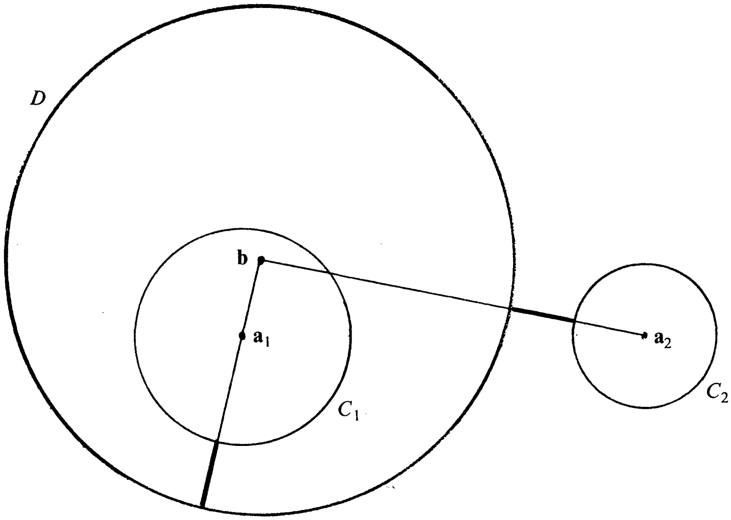


Figure 9. Another circle equidistant from C_1 and C_2.

That is, the centers of all such circles D of radius R are on an ellipse with foci $\mathbf{a}_1$ and $\mathbf{a}_2$ and sum of lengths $2R + r_2 - r_1$.

(6) Circles external to C_1 and containing C_2. Similar to (5).

(7) Circles D containing both C_1 and C_2. Let the reader show that the centers of such circles are on the other branch of the same hyperbola as in (4). The reader should also verify that there are no further cases.

One can easily make up many more challenging locus problems of these sorts. Or add additional twists. For instance, a referee has suggested asking all the same questions again with different definitions of distance.

Literature: the author is unaware of any previous literature on the whole class of locus problems discussed here—those which arise from the definition of distance between general sets. For more traditional locus problems, consult any "classical" text in high school or college geometry, e.g., [1, Section I.2] or [2, Ch. 18].

REFERENCES

1. N. Altshiller-Court, College Geometry, Johnson Publ., Richmond, Va., 1925.
2. A. W. Weeks and J. B. Adkins, A Course in Geometry, Ginn & Co., Boston, 1961.