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Linda Chen

Swarthmore College, lchen@swarthmore.edu

A. Gibney

D. Krashen

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POINTED TREES OF PROJECTIVE SPACES

L. CHEN, A. GIBNEY, AND D. KRASHEN

ABSTRACT. We introduce a smooth projective variety $T_{d,n}$ which compactifies the space of configurations of n distinct points on affine d -space modulo translation and homothety. The points in the boundary correspond to n -pointed stable rooted trees of d -dimensional projective spaces, which for $d = 1$, are $(n + 1)$ -pointed stable rational curves. In particular, $T_{1,n}$ is isomorphic to $\overline{M}_{0,n+1}$, the moduli space of such curves. The variety $T_{d,n}$ shares many properties with $\overline{M}_{0,n}$. For example, as we prove, the boundary is a smooth normal crossings divisor whose components are products of $T_{d,i}$ for $i < n$, it has an inductive construction analogous to but differing from Keel's for $\overline{M}_{0,n}$ which can be used to describe its Chow groups, Chow motive and Poincaré polynomials, generalizing [Kee92, Man95]. We give a presentation of the Chow rings of $T_{d,n}$, exhibit explicit dual bases for the dimension 1 and codimension 1 cycles. The variety $T_{d,n}$ is embedded in the Fulton-MacPherson spaces $X[n]$ for *any* smooth variety X and we use this connection in a number of ways. For example, to give a family of ample divisors on $T_{d,n}$ and to give an inductive presentation of the Chow groups and the Chow motive of $X[n]$ analogous to Keel's presentation for $\overline{M}_{0,n}$, solving a problem posed by Fulton and MacPherson.

1. INTRODUCTION

Fix an arbitrary ground field k . By a variety over k we mean a reduced (but not necessarily integral), separated scheme of finite type over k .

Let $TH_{d,n}$ denote the space of configurations of n distinct points on affine d -space up to translation and homothety. Equivalently, this may be regarded as the space of embeddings of a hyperplane and n distinct points not lying on the hyperplane in projective d -space, up to projective automorphisms. When $d = 1$, this is the moduli space $M_{0,n+1}$ of pointed rational curves. In this paper, we introduce and study varieties $T_{d,n}$ which compactify $TH_{d,n}$ for $d \geq 1, n \geq 2$. We prove that $T_{d,n}$ is a smooth, projective, irreducible, rational variety of dimension $dn - d - 1$ (c.f. Corollary 3.4.2). The points of $T_{d,n}$ are

in one to one correspondence with *stable n -pointed rooted trees* of d -dimensional projective spaces (Definition 2.0.2, Theorem 3.4.4). These pointed trees of projective spaces are higher-dimensional analogs of stable pointed rational curves. Indeed, $T_{1,n} \cong \overline{M}_{0,n+1}$ (Theorem 3.4.3). Remarkably, $T_{d,n}$ seems to share nearly all of the combinatorial and structural advantages of $\overline{M}_{0,n}$.

There has been much interest in possible higher-dimensional generalizations of $\overline{M}_{0,n}$. For example, Kapranov's Chow quotients compactify the moduli spaces of ordered n -tuples of hyperplanes in \mathbb{P}^r in general linear position in [Kap93a]. These are isomorphic to $\overline{M}_{0,n}$ when $r = 1$. Hacking, Keel, and Tevelev defined and studied another compactification of hyperplane arrangements in projective space as the closure inside the moduli spaces of stable pairs (see [KT, HKT]). These spaces are reducible, with Kapranov's Chow quotients as one component.

The Fulton-MacPherson space $X[n]$, defined as a compactification of the space of configurations of n distinct points on a smooth variety X , is also a kind of higher-dimensional analog of $\overline{M}_{0,n}$. In particular, $\mathbb{P}^1[n]$ is birational, although not isomorphic, to $\overline{M}_{0,n+3}$ (see [FM94]).

From a different perspective, one may show that $\overline{M}_{0,n+1}$ may be viewed as a subscheme of $X[n]$ for any smooth curve X . In more generality, $T_{d,n}$ arises as a subscheme of $X[n]$ for *any* smooth variety X of dimension d .

Moduli spaces of pointed rational curves are intimately related to moduli spaces of curves of higher genus g . Namely, by attaching curves in various ways, one can define maps from $\overline{M}_{0,n+1}$ to the boundary of $\overline{M}_{g,m}$. It has been possible to reduce important questions about the birational geometry of $\overline{M}_{g,n}$ for $g > 0$ to moduli of pointed rational curves [GKM02]. Analogously, by using various attaching maps, the variety $T_{d,n}$ maps to the boundary of the Fulton-MacPherson space $X[n]$. In fact, $T_{d,n}$ is a fiber of the natural projection map from the boundary component $D(N) \subset X[n]$ to X (Definition 3.1.1). We exploit this fundamental fact to study $T_{d,n}$ as well as to answer a question posed by Fulton and MacPherson about the Chow groups of $X[n]$ (cf. Theorem 4.2.1).

Another generalization of the moduli space of curves is the stack $\overline{M}_{g,n}(X, \beta)$ of stable maps from n -pointed stable curves of genus g to a variety X . When X is a point and $g = 0$, we recover $\overline{M}_{0,n}$. Stable maps have been particularly studied because their Chow rings determine the Gromov-Witten invariants of the variety X . The work of Oprea and Pandharipande, for example, show that the combinatorial structure of $\overline{M}_{0,n}$ plays a major role in the understanding the intersection theory

of the more general spaces ([Opr04a, Opr04b, Opr04c, Pan99]). It is conceivable that the $T_{d,n}$ could be used to study moduli of stable maps from higher dimensional varieties.

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1.1. Summary of Results.

Inductive construction, Chow groups, Chow motives, Poincaré polynomials, and functor of points

We describe a functor which represents $T_{d,n}$ in Section 3 (cf. Proposition 3.6.1 and Lemma 3.6.4) and use it to prove that $T_{d,n}$ can be constructed inductively. More specifically, we prove:

Theorem (3.3.1). *The variety $T_{d,n}$ may be constructed as the result of a sequence of blowups of a projective bundle over $T_{d,n-1}$.*

In particular, this gives a construction of $\overline{M}_{0,n}$ which differs from previous constructions of Keel and of Kapranov ([Kee92, Kap93b]). We use this construction to obtain an inductive presentation the Chow groups and the Chow motive of $T_{d,n}$ (Section 4) and a description of its Poincaré polynomial (Section 5).

Ample divisors

Using the embedding of $T_{d,n}$ as a closed subvariety of the Fulton-MacPherson configuration space $X[n]$ for a smooth d -dimensional variety X , we exhibit a family of ample divisors for $T_{d,n}$ in Theorem 3.5.1.

Boundary of $T_{d,n}$, and stratification

We characterize the boundary of the $T_{d,n}$, showing it is composed of smooth normal crossings divisors which are (isomorphic to) products of smaller $T_{d,i}$. In particular, for each $S \subsetneq N$, there is a nonsingular divisor $T_{d,n}(S) \subset T_{d,n}$ such that the union of these divisors $T_{d,n}(S)$ forms the boundary $T_{d,n} \setminus TH_{d,n}$. Any set of these divisors meets transversally. An intersection of divisors

$$T_{d,n}(S_1) \cap \cdots \cap T_{d,n}(S_r)$$

is nonempty exactly when the sets S_i are *nested* ; each pair is either disjoint, or one is contained in the other. Moreover, the boundary components $T_{d,n}(S)$ are products. Namely,

$$T_{d,n}(S) \cong T_{d,n-|S|+1} \times T_{d,|S|},$$

for $S \subsetneq N$, $|S| > 1$ (Theorem 3.3.1, part 4).

More generally, as in the case $d = 1$, the $T_{d,n}$ are stratified by the (closure) of the locus of points corresponding to varieties having k distinct components. There is a natural divisor class δ_N on $T_{d,n}$; for $d = 1$, $\delta_N = -\psi_{n+1}$ (beginning of Section 6). We give a simple presentation for the Chow ring of $T_{d,n}$ in terms of the δ_N and the boundary classes:

$$A^*(T_{d,n}) \cong \mathbb{Z}[\{\delta_S\}_{S \subset N, 2 \leq |S|}] / I_{d,n},$$

where the ideal $I_{d,n}$ is generated by two simple types of relations (Theorem 6.0.4). As in the case of $T_{1,n} \cong \overline{M}_{0,n+1}$, there are natural maps between the spaces given by dropping points. That is, for every $i \in \{1, \dots, n\}$, there is a natural projection $T_{d,n} \rightarrow T_{d,n-1}$ given by “dropping the i th point,” (Remark 3.6.6). We prove that $T_{d,n}$ is an HI space (Corollary 7.3.4) when defined over \mathbb{C} . That is, $H^{2*}(T_{d,n}) \cong A^*(T_{d,n})$.

We describe a relationship between boundary divisors and certain explicitly given one-cycles which yields an integer pairing between divisors and curves on $T_{d,n}$ (Theorem 6.0.5). These classes form a basis for 1-cycles modulo rational equivalence (Corollary 6.0.6). We also give an explicit conjectural pairing between cycles of complementary dimension on $T_{d,n}$ (cf. Section 6.1).

2. THE CLOSED POINTS OF $T_{d,n}$

In this section, we give a geometric description of the closed points of $T_{d,n}$ in terms of isomorphism classes of n -pointed rooted trees of d -dimensional projective spaces.

Choose a pair of smooth d -dimensional varieties X_1, X_2 , a point $p \in X_1$ and a subvariety $H \subset X_2$ such that $H \cong \mathbb{P}^{d-1}$. Let Y be the blowup of X_1 at p . We may form a new variety, which we will denote by $X_1 \#_{p,H} X_2$ by identifying the exceptional divisor in Y with the subvariety $H \in X_2$. In the case $d = 1$ this corresponds to attaching two curves together by identifying a point on one with a point on the other.

To describe a tree of d -dimensional projective spaces, we will use trees as book keeping devices. Recall that a rooted tree is a graph without cycles and with a distinguished vertex. We will use the notation $G = (V_G, E_G, v_G)$ where V_G is a set of vertices, $v_G \in V_G$ is a distinguished vertex called the root and $E_G \subset V_G \times V_G$ is the set of edges to denote such an object. Recall that given a rooted tree G , there is a natural partial order on V_G in which the root v_G is the initial or smallest element. Given $w < w'$ we say that w' is a descendant of w . In the case that $w < w'$ and there is no vertex w'' with $w < w'' < w'$, we say that w' is a daughter of w and that w is the parent of w' .

We define d -dimensional gluing data for a tree G to be a collection of projective spaces $X_w \cong \mathbb{P}^d$ for each $w \in V_G$ together with a rule which associates to each vertex $w \in V_G$ a hyperplane $H_w \subset X_w$ and to each pair $w, w' \in V_G$, where w' is a daughter of w , a point $p(w, w') \in X_w$ such that the points $p(w, w') \in X_w$ are all distinct as w' varies over the daughters of w , and do not lie on the hyperplane $H(w)$. We denote this data by (X, p, H) . Given such gluing data, we may define a variety $\#_{p,H} X$ inductively on the order of V as follows:

- (1) If $|V| = 1$, then $\#_{p,H} X = X_v = \mathbb{P}^d$,
- (2) If $|V| = n + 1$, choose a vertex $w \in V_G$ with no daughters, and let $w' \in V_G$ be the parent of w . Let G' be the tree obtained by removing w and all edges incident with w . Let X', p', H' be the restrictions of the functions X, p, H to G' . Then

$$\#_{p,H} X = \left(\#_{p',H'} X' \right) \#_{p(w',w), H_w} X_w.$$

We note that in the variety $\tilde{X} = \#_{p,H} X$, each component is a blowup of one of the varieties X_w and consequently there is a 1-1 correspondence between the components of \tilde{X} and the vertices V_G . The singular locus of \tilde{X} is exactly the intersections of the different components. Each component has a distinguished hyperplane H_w , which is in the singular locus of \tilde{X} unless $w = v_G$ is the root of G .

Definition 2.0.1. A *rooted tree of d -dimensional projective spaces* (a *d -RTPS*) is a connected variety Z together with a closed embedding $f : \mathbb{P}^{d-1} \hookrightarrow Z$ (called the *root*) such that there is some rooted tree G and gluing data (X, p, H) such that $Z \cong \#_{p,H} X$, and f defines an isomorphism of \mathbb{P}^{d-1} with H_{v_G} .

Definition 2.0.2. An *n -pointed d -RTPS* ($\mathbb{P}^{d-1} \hookrightarrow Z, p_1, \dots, p_n$) is a d -RTPS with distinct *marked points* $p_1, \dots, p_n \in Z$ such that:

- (1) p_i is not in the singular locus of Z ,
- (2) For all i , p_i does not lie in (the image of) the root.

Definition 2.0.3. ($\mathbb{P}^{d-1} \hookrightarrow Z, p_1, \dots, p_n$) is *stable* if each component $W \subset Z$ contains at least two distinct *markings*, where a marking is either a marked point p_i or an exceptional divisor. Note that each exceptional divisor corresponds to a daughter of the vertex corresponding to W .

Note that this agrees with the situation for a stable pointed rational curve. Although the standard definition in this case requires 3 markings, in our general situation we do not count the hyperplanes H_w as markings. Since each component has exactly one such hyperplane, this shows that our definition specializes to the standard one.

Definition 2.0.4. Two n -pointed rooted trees of d -dimensional projective spaces $(\mathbb{P}^{d-1} \hookrightarrow Z, p_1, \dots, p_n)$ and $(\mathbb{P}^{d-1} \hookrightarrow Z', q_1, \dots, q_n)$ are *isomorphic* if there is an isomorphism of algebraic varieties $f : Z \rightarrow Z'$ such that $f(p_i) = q_i$ and the following diagram commutes:

$$\begin{array}{ccc} & \mathbb{P}^{d-1} & \\ & \swarrow & \searrow \\ Z & \xrightarrow{\quad} & Z'. \end{array}$$

The following proposition is easy to verify explicitly:

Proposition 2.0.5. *An n -pointed rooted tree of d -dimensional projective spaces is stable if and only if it has no nontrivial automorphisms.*

As a simple example, consider a stable $(n+1)$ -pointed rational curve consisting of three components $Z_i \cong \mathbb{P}^1$ such that Z_2 and Z_3 are attached to Z_1 by identifying for each $i \in \{2, 3\}$ a point $h_i \in X_i$ with $e_i \in X_1$. Suppose there are $s \geq 2$ points $p_1, \dots, p_s \in X_2 \setminus \{h_2\}$ and $n-s \geq 2$ points $p_{s+1}, \dots, p_n \in X_3 \setminus \{h_3\}$ and the $(n+1)$ -st point p_{n+1} is on $Z_3 \setminus \{e_2, e_3\}$. The curve is a tree of projective lines and is illustrated below in figure 1. We call $p_{n+1} \in X_1$ the root of the tree.

More generally, let $Z_1 = Bl_{\{q, q'\}} \mathbb{P}^d$ be the blow up of \mathbb{P}^d at the points q and q' with exceptional divisors E_2 and E_3 , and let $H_1 = \mathbb{P}^{d-1} \hookrightarrow Z_1 \setminus \{q, q'\}$ be an embedded hyperplane. Let Z_2 be isomorphic to \mathbb{P}^d with fixed marked points $p_1, \dots, p_s \in Z_2$ and a fixed embedded hyperplane $H_2 = \mathbb{P}^{d-1} \hookrightarrow Z_2 \setminus \{p_1, \dots, p_s\}$. Finally, let Z_3 be isomorphic to \mathbb{P}^d with marked points $p_{s+1}, \dots, p_n \in Z_3$ and an embedded hyperplane $H_3 = \mathbb{P}^{d-1} \hookrightarrow Z_3 \setminus \{p_{s+1}, \dots, p_n\}$. Let H_2 be identified pointwise with E_2 and H_3 with E_3 . When the components are attached, they form a tree. We call the embedded hyperplane $H_1 = \mathbb{P}^{d-1} \subset X_1$ the root of the tree, shown below in figure 2 when $d = 2$. If $s \geq 2$ and $n-s \geq 2$, the tree is stable; it has no nontrivial automorphisms fixing the embedded hyperplanes pointwise which preserve the marked points.

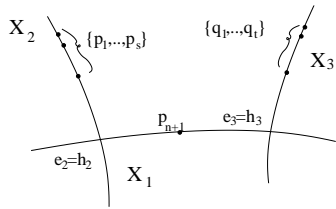


FIGURE 1. $d = 1$

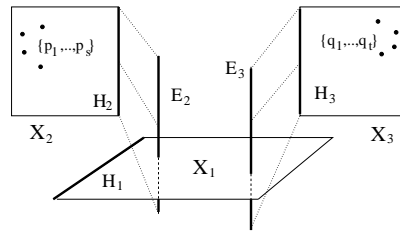


FIGURE 2. $d = 2$

3. DEFINITION AND INDUCTIVE CONSTRUCTION OF $T_{d,n}$

In this section we define the variety $T_{d,n}$ as an abstract variety. This is done by using the construction of the Fulton-MacPherson configuration space as in [FM94].

3.1. Inductive Construction of the Fulton-Macpherson Space.

Let X be a smooth variety of dimension d , and let $x \in X(k)$. As in [FM94], we let $X[n]$ denote the Fulton-MacPherson configuration space of n points on X , whose construction we recall below (with some minor changes in notation). Set $N = \{1, \dots, n\}$. The space $X[n]$ comes with a morphism $X[n] \rightarrow X^n$. For every subset $S \subset N$ with $|S| \geq 2$, Fulton and MacPherson define a codimension 1 smooth subvariety $D(S) \subset X[n]$ which maps into the diagonal $\Delta_S = \{(x_i) \in X^n | x_i = x_j \text{ for } i, j \in S\}$. In particular, we have a morphism $\pi : D(N) \rightarrow X \cong \Delta_N \subset X^n$.

Definition 3.1.1. $T_{d,n}^{X,x} = \pi^{-1}(x)$.

We shall prove that this definition does not depend on the smooth variety X or on the point $x \in X(k)$. Thus we simply write $T_{d,n}$ for $T_{d,n}^{X,x}$. To show this, we describe the functor which it represents later in this section and show that this functor is independent of our choices (see Definition 3.6.1). We also show that the points of $T_{d,n}$ correspond to the n -pointed stable d -RTPS's from the previous section (Theorem 3.4.4).

In order to set notation and motivate our work on the spaces $T_{d,n}$, we now recall Fulton and MacPherson's construction of $X[n]$.

The construction of these spaces is given inductively. It will be notationally convenient to let $N = \{1, \dots, n\}$, and we may occasionally write $X[N]$ to mean $X[n]$. For a subset $S \subset N$ we let S^+ be the subset $S \cup \{n+1\} \subset \{1, \dots, n+1\}$. In particular, $N^+ = \{1, \dots, n+1\}$.

At the n 'th step in the process, we will have constructed:

- (1) a space $X[n]$,

- (2) a morphism $\pi_n : X[n] \rightarrow X^n$ (which we will write as π when n is understood),
- (3) for each subset $S \subset N$ with $|S| \geq 2$, a divisor $D(S) \subset X[n]$.

We begin by giving the definition of the first two spaces directly. For $n = 1$, we set $X[1] = X$. For $n = 2$, we let $X[2] = Bl_\Delta(X \times X)$ be the blowup of $X \times X$ along the diagonal Δ . We define $D(\{1, 2\})$ to be the exceptional divisor of this blowup, and let $\pi : X[2] \rightarrow X^2$ be the blowup map.

To go from $X[n]$ to $X[n+1]$ requires a series of steps in itself. We'll construct a sequence of smooth varieties:

$$X[n, n] \xrightarrow{\rho^n} X[n, n-1] \xrightarrow{\rho^{n-1}} \dots \longrightarrow X[n, k+1] \xrightarrow{\rho^{k+1}} X[n, k] \longrightarrow \dots \longrightarrow X[n, 1] \xrightarrow{\rho^1} X[n, 0],$$

so that $X[n, n] = X[n+1]$. We will define these varieties $X[n, k]$ inductively with respect to k . At each step, we will construct for $0 \leq k \leq n$:

- (1) a smooth variety $X[n, k]$,
- (2) a morphism $\rho_k : X[n, k] \rightarrow X[n, k-1]$ when $k > 0$,
- (3) smooth subvarieties $X[n, k](S')$ for each subset $S' \subset N^+$ with at least two elements.

In the case $k = 0$, we set $X[n, 0] = X[n] \times X$. For $S' \subset N \subset N^+$, we define $X[n, 0](S') = D(S') \times X$. Let $p_i : X[n] \rightarrow X^n \rightarrow X$ be the composition of π_n with the i 'th projection map. We define $X[n, 0](\{i\}^+)$ to be the graph of p_i - in other words, it is the image of the morphism $id \times p_i : X[n] \rightarrow X[n] \times X$. Now suppose that $S \subset N$, $|S| \geq 2$. It turns out that if $i, j \in S$, then if we let $\Gamma_i = id \times p_i : D(S) \rightarrow D(S) \times X$, the images $\Gamma_i(D(S))$ and $\Gamma_j(D(S))$ are isomorphic. We denote these common maps by Γ_S and define $X[n, 0](S^+)$ to be the image $\Gamma_S(D(S))$.

The variety $X[n, 1]$ is defined to be the blowup of $X[n, 0]$ along the subvariety $X[n, 0](N^+)$. For $S' \neq N^+$, the variety $X[n, 1](S')$ is defined to be the proper transform of $X[n, 0](S')$, and we define $X[n, 1](N^+)$ to be the exceptional divisor. We have the following pullback diagram:

$$\begin{array}{ccc} X[n, 1](N^+) & \xlongequal{\quad} P(\mathcal{N}_N) & \longrightarrow X[n, 1] \\ & \downarrow & \downarrow \rho_1 \\ X[n, 0](N^+) & \longrightarrow X[n, 0] & \xlongequal{\quad} X[n] \times X, \end{array}$$

where $\mathcal{N}_N = N_{X[n, 0](N^+)} X[n, 0]$ is the normal bundle of $X[n, 0](N^+)$ in $X[n, 0]$ and $X[n, 1](N^+) = P(\mathcal{N}_N)$ is the exceptional divisor of the

blowup. It will be useful to consider the first Chern class of this bundle, and so we will set $l_N = c_1(\mathcal{O}_{\mathcal{N}_N}(1))$.

Once $X[n, k]$ has been constructed for $k \geq 1$ along with its subschemes $X[n, k](S')$, the variety $X[n, k+1]$, together with its morphism $\rho_{k+1} : X[n, k+1] \rightarrow X[n, k]$, is defined to be the blowup along the disjoint union of the subvarieties $X[n, k](U^+)$, where U ranges over all subsets of N of cardinality $n - k$. Fulton and MacPherson prove that these subvarieties are all disjoint ([FM94]). For $S' = U^+$ where $U \subset N$, $|U| = n - k$, we define $X[n, k+1](U^+)$ to be the exceptional divisor in $X[n, k+1]$ lying over $X[n, k](U^+)$. For $S' \in N^+$ not of this form, we define $X[n, k+1](S')$ to be the proper transform of $X[n, k](S')$.

For each $U \subset N$ of cardinality $|U| = n - k$ we have the following pullback diagram:

$$\begin{array}{ccc} X[n, k+1](U^+) & \xlongequal{\quad} & P(\mathcal{N}_U) & \longrightarrow & X[n, k+1] \\ & & \downarrow & & \downarrow \\ & & X[n, k](U^+) & \longrightarrow & X[n, k], \end{array}$$

where $\mathcal{N}_U = N_{X[n, k](U^+)}X[n, k]$ is the normal bundle of $X[n, k](U^+)$ in $X[n, k]$ and $X[n, k+1](U^+) = P(\mathcal{N}_U)$ is the exceptional divisor of the blowup. We write $l_U = c_1(\mathcal{O}_{\mathcal{N}_U}(1))$.

To complete the construction of $X[n+1] = X[n, n]$, we define for $S' \in N^+$, $X[n+1](S) = X[n, n](S')$ and $\pi : X[n+1] \rightarrow X^n$ to be the composition

$$X[n+1] = X[n, n] \xrightarrow{\rho_1 \circ \dots \circ \rho_n} X[n, 0] = X[n] \times X \xrightarrow{\pi_n \times id_X} X^n$$

For convenience of notation, we define $X[n, i](S_1, \dots, S_k) = X[n, i](S_1) \cap \dots \cap X[n, i](S_k)$.

Theorem 3.1.2. *Let $\emptyset \neq S \subset N \neq 2$, $|S| = i$. Choose $a \in N$. Then:*

- (1) *The morphisms $X[n, n-1](\{a\}^+) \rightarrow X[n]$ and $X[n, n-s](S^+) \rightarrow X[n, 0](S^+) \cong D(S)$ are isomorphisms for $|S| \geq 2$.*
- (2) *The morphism $X[n, 1](N^+) \rightarrow X[n, 0](N^+)$ is a projective bundle morphism of relative dimension d .*
- (3) *The morphism $X[n, i+1](N^+) \rightarrow X[n, i](N^+)$ for $1 \leq i \leq n-1$ is a blowup along the union of subvarieties $X[n, i](N^+, S^+)$ for $S \subset N$, $|S| = n - i$.*
- (4) *For S as above, the morphism $X[n, i](N^+, S^+) \rightarrow X[n, 0](N^+, S^+)$ is an isomorphism.*

Proof. Parts 1, 2, and 3 follow from [FM94] proposition 3.5. For part 4, we have by part 1, $X[n, i](S^+) \rightarrow X[n, 0](S^+)$ is an isomorphism. Consequently, when we restrict to $X[n, i](N^+, S^+)$, we get an isomorphism $X[n, i](N^+, S^+) \cong X[n, 0](N^+, S^+)$. \square

3.2. Functors of points related to the Fulton-Macpherson space.

It will be useful for us to have a description of the functors represented by the varieties $X[n, i]$ and $X[n, i](S_1, \dots, S_k)$. Suppose H is a variety and $h : H \rightarrow X^{n+1}$ is a morphism. For a subset $S \subset N$, we use the notation h_S to denote the composition of h with the projection $X^{N^+} \rightarrow X^S$, and we write h_a for $h_{\{a\}}$. Let $\Delta_S \subset X^S$ denote the (small) diagonal, and \mathcal{I}_S the ideal sheaf of Δ_S in X^S . Note that for $S_1 \subset S_2$, the projection $X^{S_2} \rightarrow X^{S_1}$ induces a morphism $h_{S_1}^* \mathcal{I}_{S_1} \rightarrow h_{S_2}^* \mathcal{I}_{S_2}$.

Definition 3.2.1. Let H and h be as above. A *screen* for h and $S \subset N^+$ is an invertible quotient $\phi_S : h^* \mathcal{I}_S \rightarrow \mathcal{L}_S$. A collection of screens $\phi_{S_1}, \dots, \phi_{S_k}$ is *compatible* if whenever $S_i \subset S_j$, there is a unique morphism $\mathcal{L}_{S_i} \rightarrow \mathcal{L}_{S_j}$ which makes the following diagram commute:

$$\begin{array}{ccc} h_{S_i}^* \mathcal{I}_{S_i} & \longrightarrow & \mathcal{L}_{S_i} \\ \downarrow & & \downarrow \\ h_{S_j}^* \mathcal{I}_{S_j} & \longrightarrow & \mathcal{L}_{S_j} \end{array}$$

Definition 3.2.2. A subset $S \subset N^+$ satisfies *property P_i* , ($S \in P_i$) if either $S \subset N$, $|S| \geq 2$ or $S = T^+$, $|T| > n - i$.

Definition 3.2.3. We define the functor $\mathcal{X}[n, i]$ from the category of schemes to the category of sets by setting $\mathcal{X}[n, i](H)$ to be the set of pairs

$$\left((h : H \rightarrow X^{N^+}), \{ \phi_T : h_T^* \mathcal{I}_T \rightarrow \mathcal{L}_T \}_{T \in P_i} \right),$$

such that the ϕ_T 's form a compatible collection of screens. We define the subfunctor $\mathcal{X}[n, i](S_1, \dots, S_k)$ by setting $\mathcal{X}[n, i](S_1, \dots, S_k)(H)$ to be the subset of $\mathcal{X}[n, i]$ such that whenever $T \in P_i$ with $|T \cap S_j| \geq 2$ and $T \not\subset S_j$ for some j , the compatibility morphism $\mathcal{L}_{T \cap S_j} \rightarrow \mathcal{L}_T$ is zero.

The following theorem is useful not only for understanding the iterative construction of the space $T_{d,n}$, but also for the applications to the Fulton-MacPherson configuration space in Section 4.

Theorem 3.2.4 ([FM94]). *The functors $\mathcal{X}[n, i]$ and $\mathcal{X}[n, i](S_1, \dots, S_k)$ are represented by the varieties $X[n, i]$ and $X[n, i](S_1, \dots, S_k)$ respectively.*

3.3. Inductive construction of $T_{d,n}$. We now present an inductive construction of $T_{d,n}$ as a sequence of blowups of a projective bundle over $T_{d,n-1}$. This allows us to give an explicit inductive presentation of its Chow groups, its Chow motive and in the next section, a description of its Poincaré polynomial.

Theorem 3.3.1. *There is a sequence of smooth varieties $F_{d,n}^i$, for $0 \leq i \leq n$, with subvarieties $F_{d,n}^i(T)$ indexed by $T \subsetneq N^+$, $|T| \geq 2$ and morphisms $b_i : F_{d,n}^{i+1} \rightarrow F_{d,n}^i$ such that:*

- (1) $F_{d,n}^0 = T_{d,n}$ and $F_{d,n}^n = T_{d,n+1}$,
- (2) the morphism $b_0 : F_{d,n}^1 \rightarrow F_{d,n}^0 = T_{d,n}$ is a projective bundle morphism of relative dimension d ,
- (3) the morphisms b_i for $1 \leq i \leq n-1$ are blowups along the union of the subvarieties $F_{d,n}^i(S^+)$, $|S| = n-i$,
- (4) $F_{d,n}^i(S^+) \cong T_{d,s} \times T_{d,n-s+1}$ where $s = |S| = n-i$ if $i \neq n-1$,
- (5) $F_{d,n}^{n-1}(\{a\}^+) \cong T_{d,n}$.

This theorem closely parallels the above situation for the Fulton MacPherson configuration space $X[n+1]$ from $X[n] \times X$, and in fact we derive it mainly as a consequence of a careful analysis of certain aspects of this construction. Although very similar in structure to the construction of Keel [Kee92] in the case $d=1$, we note that our construction is different. For example, the analog of our (nontrivial) projective bundle b_0 in Keel's construction is always a trivial vector bundle.

The varieties $F_{d,n}^i$ and $F_{d,n}(S)$ are defined as follows:

Definition 3.3.2. Let X be a smooth projective d -dimensional variety, and choose $x \in X(k)$. We abuse notation slightly (by using the symbol π in two different ways) and let $\pi : X[n,i](N^+) \rightarrow X$ be the composition

$$X[n,i](N^+) \longrightarrow X[n,i] \xrightarrow{\rho_1 \cdots \rho_i} X[n] \times X \xrightarrow{\pi \times id_X} X^{n+1} \longrightarrow X$$

where the final arrow is any of the projections (all give the same result). We define $F_{d,n}^i$ to be $\pi^{-1}(x) = X[n,i](N^+) \times_X x$ and $F_{d,n}^i(S) = X[n,i](N^+, S^+) \times_X x$. We also define $T_{d,n}(S) = F_{d,n}^0(S^+) \subset F_{d,n}^0 = T_{d,n}$.

The proof of Theorem 3.3.1 follows from the following proposition combined with Theorem 3.1.2.

Proposition 3.3.3. *Suppose X is a smooth variety with trivial tangent bundle, and let $S \subset N$, $|S| = s$. Then if $0 \leq i \leq s$, we have an isomorphism $X[n, n-s+i](S^+) = X[n-s+1] \times F_{d,s}^i$ commuting with the natural projections to $X[n-s+1]$.*

In particular, in proving this proposition, it immediately follows that Definitions 3.3.2 and 3.1.1 do not depend on the variety X or on the point x , since the tangent bundle is locally trivial. Note that it follows that for X with trivial tangent bundle, we have $X[n, n-s+i](S^+, T^+) \cong X[n-s+1] \times F_{d,n}^i(T^+)$ for $T \subsetneq S$ by restricting the above isomorphism.

Remark 3.3.4. We see by this reasoning that in definition 3.3.2, the maps $\pi : X[n, i](N^+) \rightarrow X$ and $\pi|_{X[n, i](N^+, S^+)}$ locally have a product structure as above, and so in particular, $[F_{d,n}(S)] = \iota_x^! [X[n, i](N^+, S^+)]$ in the Chow groups of $X[n, i](N^+)$ and $F_{d,n}$ respectively, where ι_x is the inclusion of the point x into X . Since the class $[X[n, i](N^+, S^+)]$ on $X[n, i](N^+)$ can also be seen as the Gysin pullback $j^! [X[n, i](S^+)]$ where $j : X[n, i](N^+) \hookrightarrow X[n, i]$ is the natural embedding, we may write $[F_{d,n}(S^+)] = \iota^! j^! [X[n, i](S^+)]$, and it makes sense therefore to abuse notation and formally define $[F_{d,n}(N^+)] = \iota^! j^! [X[n, i](N^+)]$. We also set $[T_{d,n}(S)] = [F_{d,n}^0(S^+)]$.

proof of proposition 3.3.3. By Theorem 3.2.4, we may translate a morphism to $X[n, n-s+i](S^+)$ to a collection of screen data. Suppose $f : H \rightarrow X[n, n-s+i](S^+)$ corresponds to a collection of screens

$$\left((h : H \rightarrow X^{N^+}), \{ \phi_T : h_T^* \mathcal{I}_T \rightarrow \mathcal{L}_T \}_{T \in P_i} \right).$$

Note that if $T \subset S$ or $T = S^+$, the function h_T maps entirely into the diagonal $\Delta_T \subset X^T$, and so by factoring h_T through $\Delta_T \cong X$, we may identify $h_T^* \mathcal{I}_T$ with the pullback of the conormal sheaf $h_{\{a\}}^* \mathcal{I}_T / \mathcal{I}_T^2 = ((TX)^T / TX)^*$, where $a \in T$. Note that we have $h_{\{a\}} = h_{\{b\}}$ for any $b \in S^+$, and so it suffices to choose $a \in S^+$. The screen for T therefore translates to specifying a 1-dimensional sub-vector bundle $i_T : L_T \hookrightarrow (TX)^T / TX$ compatible with respect to the natural projections $(TX)^T / TX \rightarrow (TX)^{T'} / TX$ for $T' \subset T$. Since V is trivial, we may write $TX = V_X$ where V is a vector space over k of dimension d . Let $\mathcal{F}_{d,s}(H)$ denote the set of collections of compatible line sub-vector bundles of the form $\{i_T : L_T \hookrightarrow V^T / V\}_{T \subset S \text{ or } T=S^+}$. It is not hard to check from this description that $\mathcal{F}_{d,s}$ in fact defines a functor which is represented by $F_{d,s}$.

To complete the proof, we note that by the above, a morphism to $X[n, n-s+i](S^+)$ gives a morphism to $F_{d,s}^i$. We may also obtain a morphism to $X[n-s+1]$ by choosing a point $a \in S^+$ and keeping only the screens for subsets $T \subset (N \setminus S) \cup \{a\}$. This gives a morphism $X[n, n-s+i](S^+) \rightarrow X[n-s+1] \times F_{d,s}$. We may obtain an inverse morphism by describing how to take the screens for $T \subset (N \setminus S) \cup \{a\}$ together with the screens for $T \subset S^+$ and define screens for the

remaining $T \in P_{n-s+1}$. This is done in a way similar to the proof of Theorem 3.1.2, part 1 and we leave the detailed proof to the reader. We mention as a guide that for T of the form $T = U \cup R$ where $U \subset S^+$ and $R \cap S^+ = \emptyset$, we set $\mathcal{L}_T = \mathcal{L}_{R \cup \{a\}}$ if $R \neq \emptyset$, and otherwise $\mathcal{L}_T = L_T^*$ if $T \subset S^+$. \square

proof of Theorem 3.3.1. Note that we may obtain the product decomposition $T_{d,n}(S) \cong T_{d,s} \times T_{d,n-s+1}$, where $s = |S|$ by letting X be a variety with trivial tangent bundle and considering the commutative diagram where the upper square is a pullback:

$$\begin{array}{ccc}
X[n, n-s+i](S^+, N^+) & \longrightarrow & X[n, n-s+i](S^+) \\
\text{fibers } \cong F_{d,s}^i \downarrow & & \downarrow \text{fibers } \cong F_{d,s}^i \\
X[n-s+i, 0] \left(((N \setminus S) \cup \{a\})^+ \right) & \longrightarrow & X[n-s+1] \\
\text{fibers } \cong T_{d,n-s+1} \downarrow & & \downarrow \\
X & \xlongequal{\quad\quad\quad} & X
\end{array}$$

\square

3.4. Basic properties of $T_{d,n}$. We now give an example of these spaces for “minimal” values of n :

Proposition 3.4.1. *We have isomorphisms $T_{1,3} \cong \mathbb{P}^1$ and for $d > 1$, $T_{d,2} \cong \mathbb{P}^{d-1}$. Under these identifications, $[T_{1,3}(\{1, 2, 3\})] = \mathcal{O}_{\mathbb{P}^1}(-1)$, and $[T_{d,2}(\{1, 2\})] = \mathcal{O}_{\mathbb{P}^{d-1}}(-1)$.*

Proof. First consider the case of $T_{d,2}$. We know that $\mathbb{A}^d[2] = Bl_{\Delta}(\mathbb{A}^d \times \mathbb{A}^d)$, and that $D(\{1, 2\})$ is the exceptional divisor of the blowup [FM94]. Therefore

$$D(\{1, 2\}) = P(N_{\Delta}(\mathbb{A}^d \times \mathbb{A}^d)) \cong \mathbb{P}^{d-1} \times \mathbb{A}^d.$$

In particular, $T_{d,2} \cong \mathbb{P}^{d-1}$ as claimed.

For the case of $T_{1,3}$, we note that $\mathbb{A}^1[3] = Bl_{\Delta}(\mathbb{A}^1)^3$, where Δ is the small diagonal with exceptional divisor $D(\{1, 2, 3\})$. \square

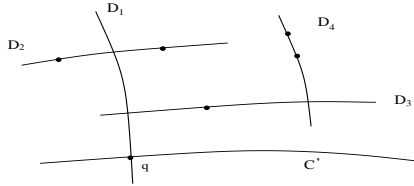
Corollary 3.4.2. *$T_{d,n}$ is a smooth projective rational variety of dimension $dn - d - 1$.*

Proof. This follows from induction on n , with base case proven in Proposition 3.4.1 and inductive step given by Theorem 3.3.1. \square

Proposition 3.4.3. *$T_{1,n} \cong \overline{M}_{0,n+1}$.*

Proof. Since each n pointed stable 1-RTPS is exactly an $(n+1)$ -pointed stable curve (where the root hyperplane is identified with the $(n+1)$ st marking), the family $T_{d,n}^+ \rightarrow T_{d,n}$ gives a morphism $T_{d,n} \rightarrow \overline{M}_{0,n}$ which is bijective on \overline{k} points. To show this is an isomorphism, it suffices to construct an inverse morphism.

To do this, consider the Fulton-MacPherson configuration space $\mathbb{P}^1[n]$, and the divisor $D(N)$ on $\mathbb{P}^1[n]$. By Definition 3.1.1, we may identify $T_{1,n}$ with a fiber of the natural morphism $D(N) \rightarrow \mathbb{P}^1$. By [MM] (pages 4,5), there is a natural isomorphism $\overline{M}_{0,n}(\mathbb{P}^1, 1) \cong \mathbb{P}^1[n]$. Note that by [FP] (Theorem 2, part 3), $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is actually a fine moduli space for stable maps of degree 1 to \mathbb{P}^1 . The isomorphism from [MM] identifies the divisor $D(N)$ with the stable maps $f : (C, p_1, \dots, p_n) \rightarrow \mathbb{P}^1$ which take all the marked points to a given point $x \in \mathbb{P}^1$. The natural morphism $D(N) \rightarrow \mathbb{P}^1$ is simply the morphism taking this stable map to x . For such a stable map, the semistable curve C must have the form:



Since $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is a fine moduli space, one may check that the fiber over a given point $x \in \mathbb{P}^1$ is also a fine moduli space for $(n+1)$ -pointed stable curves. To see this, suppose we have a stable map $f : (C, p_1, \dots, p_n) \rightarrow \mathbb{P}^1$ in this fiber, where the semistable curve C has irreducible components C', D_1, \dots, D_r , with $f_*([C']) = 1$, $f_*([D_i]) = 0$. Since none of the marked points p_i may lie on C' , the curve remaining after forgetting C' , composed of the union of the D_i , is a $(n+1)$ -pointed stable rational curve where the $(n+1)$ st marking is obtained from the point where $\cup D_i$ intersects C' . This gives an isomorphism of the fiber over x with $\overline{M}_{0,n+1}$ which is inverse to the morphism above. \square

Theorem 3.4.4. *The points of $T_{d,n}$ are in one to one correspondence with isomorphism classes of n -pointed stable rooted trees of d -dimensional projective spaces.*

Proof. Let $X[n]^+ \rightarrow X[n]$ be the “universal family” as described in [FM94]. By base change over $T_{d,n} \rightarrow X[n]$ (included by choosing a point $x \in X$), we obtain a flat family $T_{d,n}^+ \rightarrow T_{d,n}$. It follows from the description in [FM94] that the fibers are all n pointed stable d -RTPS’s, exactly one in each isomorphism class. \square

3.5. **Ample divisor classes on $T_{d,n}$.** Although by 3.4.2, we know abstractly that $T_{d,n}$ is a projective variety, it is often useful to have an explicit presentation of an ample divisor. We exhibit such a divisor below:

Theorem 3.5.1. *Let $\delta_S = [T_{d,n}(S)]$ in the Chow group of $T_{d,n}$ as in remark 3.3.4. Then for $S \subset N, |S| \geq 2$, the divisor classes*

$$\eta_S = \sum_{N \supseteq T \supseteq S} \delta_T$$

is nef and base point free. Furthermore, any expression of the form

$$A = \sum_{S \subset N, |S| \geq 2} c_S \eta_S$$

is very ample provided $c_S > 0$ for all S .

Proof. By [FM94], we have for any smooth d -dimensional variety X , an embedding

$$i : X[n] \hookrightarrow \prod_{S \subset N, |S| \geq 2} Bl_{\Delta_S}(X^S),$$

where $\Delta_S \subset X^S$ is the (small) diagonal. Let i_S be the morphism i composed with the projection map onto the factor $Bl_{\Delta_S}(X^S)$. If A_S is an very ample class on $Bl_{\Delta_S}(X^S)$, then it follows immediately that $i_S^*(A_S)$ is nef and base point free (since it is the pullback of a very ample divisor), and that $\sum c_S i_S^*(A_S)$ is very ample on $X[n]$ if each $c_S > 0$ (since it is the pullback of a very ample divisor via an embedding).

Let B'_S be a very ample divisor class on X^S , and let I_S be the ideal sheaf of Δ_S in \mathcal{O}_{X^S} . Let $f_S : Bl_{\Delta_S}(X^S) \rightarrow X^S$ be the natural projection. Then $f_S^{-1}(I_S)$ is an invertible sheaf and the divisor class $\lambda f_S^* B'_S + c_1(f_S^{-1}(I_S))$ is very ample for some $\lambda > 0$ (see [Har77], Proposition 7.10(b) and the proof of 7.13(a)). Let $B_S = \lambda B'_S$. Then

$$\alpha_S = i_S^* \left(f_S^* B_S + c_1(f_S^{-1}(I_S)) \right)$$

is nef and base point free. Fix for the remainder of the proof integers $c_S > 0$. Then

$$\alpha = \sum_{S \subset N, |S| \geq 2} i_S^* \left(f_S^* B_S + c_1(f_S^{-1}(I_S)) \right)$$

is very ample on $X[n]$.

Now consider the embedding $T_{d,n} \xrightarrow{j} X[n]$, and let j_S be the composition with the projection to $Bl_{\Delta_S}(X^S)$. Let $\eta_S = j^* \alpha_S$, and $A = j^* \alpha$. Since j is an embedding, η_S is nef and base point free and A is very

ample. We will be done once we show that the divisors η_S have the desired form. To begin, we may rewrite η_S in the following way:

$$\begin{aligned}\eta_S &= j^* \alpha_S = f_S^* B_S + c_1(f_S^{-1}(I_S)) \\ &= j^* i_S^* f_S^* B_S + j^* c_1(i_S^* f_S^{-1}(I_S)) \\ &= (f_S \circ i_S \circ j)^* B_S - j^* c_1(i_S^* f_S^{-1}(I_S))\end{aligned}$$

Examining the second term, we see by lemma 7.2.1, that $i_S^* f_S^{-1}(I_S) = (f_S \circ i_S)^{-1}(I_S)$, and by [FM94], page 203, if we let $I(D(S))$ be the ideal sheaf of $D(S) \subset X[n]$, we have $(f_S \circ i_S)^{-1}(I_S) = \prod_{T \supset S} I(D(T))$. Consequently, taking first Chern classes and applying j^* , we have:

$$j^* c_1(i_S^* f_S^{-1}(I_S)) = j^* \sum_{T \supset S} c_1(I(D(T))) = j^* \sum_{T \supset S} -[D(T)] = - \sum_{T \supset S} \delta_T.$$

On the other hand, looking at the first term, $(f_S \circ i_S \circ j)^* B_S$, we see that since the morphism $(f_S \circ i_S \circ j)$ factors through a morphism to $\text{Spec}(k)$, whose Picard group is the zero group, this pullback must in fact vanish. Therefore we have

$$\eta_S = \sum_{S \subset T \subset N} \delta_T,$$

as desired. □

3.6. A relative version of $T_{d,n}$. It will be useful in what follows to have a relative version of construction of $T_{d,n}$. This follows without much technical difficulty and we leave some of the routine verifications to the reader.

Definition 3.6.1. Let V be a rank d vector bundle over a scheme X . We define the functor $\mathcal{T}_{V,n}$ from the category $(\text{Sch}/X)^{op}$ of X -schemes to the category of sets as follows. For $h : H \rightarrow X$, we define $\mathcal{T}_{X,n}(H)$ to be the set of collections $\{\phi_T : h^* V \rightarrow \mathcal{L}_T\}_{T \in N, |T| \geq 2}$, such that the ϕ_T 's form a compatible collection of screens (in the same sense as in 3.2.1). Note that there is a canonical morphism (natural transformation) from $\mathcal{T}_{V,n}$ to X .

By setting $T_{V,1} = X$, $T_{V,2} = \text{Proj}_X(\text{Sym}^\bullet V)$, we may inductively define varieties $T_{V,n}$, $F_{V,n}^i$, $F_{V,n}(S^+)$ such that the following theorem holds:

Theorem 3.6.2. *There is a sequence of schemes $F_{V,n}^i$, for $0 \leq i \leq n$, with subschemes $F_{V,n}^i(T)$ indexed by $T \subsetneq N^+$, $|T| \geq 2$ and morphisms $b_i : F_{V,n}^{i+1} \rightarrow F_{V,n}^i$ such that:*

- (1) $F_{V,n}^0 = T_{V,n}$ and $F_{V,n}^n = T_{V,n+1}$,
- (2) the morphism $b_0 : F_{V,n}^1 \rightarrow F_{V,n}^0 = T_{V,n}$ is a projective bundle morphism of relative dimension d ,
- (3) the morphisms b_i for $1 \leq i \leq n-1$ are blowups along the union of the subschemes $F_{V,n}^i(S^+)$, and $F_{V,n}^i(S^+) \cong T_{V,s} \times_X T_{V,n-s+1}$ if $i \neq n-1$ and $F_{V,n}^{n-1}(\{a\}^+) \cong T_{V,n}$,
- (4) $T_{V,n}$ is represented by $T_{V,n}$.

The proof of this theorem takes the following steps. First, we see that it holds in the case that V is a trivial bundle by noting that the inductive construction of the space $T_{d,n+1}$ from $T_{d,n}$ by taking a projective bundle and blowing up may be fibered with a scheme X to give an inductive construction of $T_{d,n+1} \times X = T_{V,n+1}$ from $T_{d,n} \times X = T_{V,n}$. For a general bundle, we may define the functors $\mathcal{F}_{V,n}^i(S^+)$ by emulating the definition of $\mathcal{X}[n, i](N^+, S^+)$, and note that they are locally represented subschemes over subsets where V is trivial, and that these subschemes glue to give a closed subscheme $F_{V,n}^i(S^+)$. We define $F_{V,n}^{i+1}$ to be the blowup along the subschemes $F_{V,n}^i(S^+)$ where $|S| = i$. Since these have the correct functorial description locally, we may glue and conclude that $F_{V,n}^{i+1}$ represents the functor $\mathcal{F}_{V,n}^{i+1}$.

We note the following lemmas which will be useful in Section 4:

Lemma 3.6.3. *Let V be a vector bundle on X , and let $\pi : T_{V,n} \rightarrow X$ be the natural projection. Then $T_{V,n} \times_X T_{V,m} = T_{\pi^*T_{V,n},m}$.*

Lemma 3.6.4. *Let X be any scheme, and let V be a trivial vector bundle of rank d on X . Then $T_{d,n} \times X \cong T_{V,n}$. In particular, if $X = \text{Spec}(k)$ then $T_{V,n} \cong T_{d,n}$.*

Lemma 3.6.5. *Let X be a smooth variety and let $D(S) \subset X[n]$ be the divisor on the Fulton-MacPherson configuration space described in the beginning of the section. If $\pi : D(S) \rightarrow X[n - |S| + 1] = X[(N \setminus S) \cup \{a\}]$ is the natural morphism from projecting with respect to the subset $(N \setminus S) \cup \{a\}$ where $a \in S$, and $f : X[(N \setminus S) \cup \{a\}] \rightarrow X$ the projection with respect to a , then there is a natural isomorphism $D(S) = T_{f^*\Omega_{X,S}}$.*

The proofs of these are elementary and follow from an examination of the functorial descriptions of the spaces involved.

Remark 3.6.6. The morphism $T_{d,n+1} \rightarrow T_{d,n}$ obtained by composing the morphisms b_i of theorem 3.6.2 is given functorially by dropping all screens for subsets $S \subset N^+$ which contain the $(n+1)$ st marking. In the future, we denote this morphism by π_{n+1} . We may similarly define morphism π_i for any $i \in N^+$ by dropping the i th marking.

4. INDUCTIVE PRESENTATIONS OF CHOW GROUPS AND MOTIVES

We consider the Chow group of a variety X as a graded abelian group $A(X) = A_*(X)$. We use the following general conventions. For a graded abelian group $M = \bigoplus_{i \in \mathbb{Z}} M_i$, we set $M(n)$ to be the group with grading shifted so that $(M(n))_i = M_{n-i}$. The grading on the tensor product $M \otimes N = M \otimes_{\mathbb{Z}} N$ is given by $(M \otimes N)_n = \bigoplus_{i+j=n} M_i \otimes N_j$. We write \mathbb{Z} for the graded abelian group with the integers in degree 0 and zero in all other degrees.

Remark 4.0.7. All the constructions used in this section are motivic: in other words, if the reader prefers, they may interpret $A(X)$ as $M(X)$, the Chow motive of X (as in [Man68]), $M(i)$ to be M twisted i times with the Lefschetz motive, and \mathbb{Z} to be the motive of $\text{Spec}(k)$.

In this notation we have the following well known facts:

Lemma 4.0.8. *For V a vector bundle of rank d on X ,*

$$A(P(V)) = \bigoplus_{i=0}^{d-1} A(X)(i) = A(X) \otimes \left(\bigoplus_{i=0}^{d-1} \mathbb{Z}(i) \right) = A(X) \otimes A(\mathbb{P}^{d-1}).$$

Proof. [Ful98] for Chow groups, [Man68] for Chow motives. □

Lemma 4.0.9. *For $Z \hookrightarrow X$ a regularly embedded subvariety of codimension d ,*

$$A(\text{Bl}_Z X) = A(X) \oplus \bigoplus_{i=1}^{d-1} A(Z)(i) = A(X) \oplus \left(A(Z) \otimes A(\mathbb{P}^{d-2}(1)) \right).$$

Proof. [Man68] □

One technical difficulty which makes the computation of Chow groups more difficult than the computation of cohomology is the fact that the Chow group of a product $X \times Y$ is not easily expressible in terms of the Chow groups of X and Y . Philosophically, we show in this section that products (and certain fiber bundles) with $T_{d,n}$ as one of the factors are not subject to this difficulty.

4.1. The Chow groups and motives of $T_{d,n}$.

Theorem 4.1.1. *Let V/X be a vector bundle of rank d . Then*

$$A(T_{V,n+1}) = \left(\bigoplus_{j=0}^d A(T_{V,n})(j) \right) \oplus \left(\bigoplus_{j=1}^d \bigoplus_{S \subsetneq N, |S| \geq 2} A(T_{V,|S|} \times T_{V,n-|S|+1})(j) \right) \oplus \left(\bigoplus_{j=1}^{d-1} \bigoplus_{a \in N} A(T_{V,n})(j) \right)$$

Proof. This is an immediate consequence of Theorem 3.6.2 together with Lemmas 4.0.8 and 4.0.9. \square

Using Lemma 3.6.4 to identify $T_{d,n} \times B = T_{\mathcal{O}_B^d, n}$ for a variety B , we obtain the following corollary:

Corollary 4.1.2. *Let V/X be a vector bundle of rank d . Then*

$$A(T_{d,n+1} \times B) = \left(\bigoplus_{j=0}^d A(T_{d,n} \times B)(j) \right) \oplus \left(\bigoplus_{j=1}^d \bigoplus_{S \subsetneq N, |S| \geq 2} A(T_{d,|S|} \times T_{d,n-|S|+1} \times B)(j) \right) \oplus \left(\bigoplus_{j=1}^{d-1} \bigoplus_{a \in N} A(T_{d,n} \times B)(j) \right)$$

In particular, setting $B = \text{Spec}(k)$, we obtain a presentation for the Chow groups of $T_{d,n+1}$ in terms of the Chow rings of varieties of the form $T_{d,t} \times B$ for various values of $t < n$. These terms may be successively reduced using corollary 4.1.2, eventually using the fact that $A(T_{1,3}) = A(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)$ or the fact $A(T_{d,2}) \cong A(\mathbb{P}^{d-1}) = \bigoplus_{i=0}^{d-1} \mathbb{Z}(i)$ from Lemma 3.4.1.

4.2. The Chow groups and motives of the Fulton-MacPherson configuration space. Fulton and MacPherson have given a compactification $X[n]$ of the moduli space of n distinct points on a smooth variety X as well as a presentation of the intersection ring of the space $X[n]$. They pose the following problem ([FM/Ann94], page 189):

”It would be interesting to find an explicit basis for the Chow groups of $\mathbf{P}^m[n]$, preferably simple with respect to the intersection pairings, as Keel has done in the case $m = 1$.”

In this section, we present an inductive presentation of the Chow groups and motives of the spaces $X[n]$ which parallels Keel’s presentation in [Kee92].

Let us begin by making an elementary observation. The blowup construction of the Fulton-MacPherson space described in Section 3 gives $X[n+1]$ as a composition of blowups of $X[n] \times X$. In the same way, we easily obtain $X[n+1] \times B$ as a composition of blowups of $X[n] \times X \times B$ for an arbitrary variety B . Together with the identifications from Theorem 3.1.2, part 1, this yields the following inductive presentation:

Theorem 4.2.1. *Let X be a smooth variety. Then*

$$A(X[n+1] \times B) = A(X[n] \times X \times B) \oplus \left(\bigoplus_{j=1}^d \bigoplus_{S \subsetneq N, |S| \geq 2} A(D(S) \times B)(j) \right) \oplus \left(\bigoplus_{j=1}^{d-1} \bigoplus_{a \in N} A(X[n] \times B)(j) \right)$$

As before, the symbol A can stand either for the Chow group or the Chow motive. In particular, setting $B = \text{Spec}(k)$, we obtain a presentation for the Chow groups (or motives) of $X[n+1]$ in terms of the Chow groups (or motives) of the varieties $D(S)$ and $X[n] \times X$. From Lemma 3.6.5, we have an isomorphism $D(S) = T_{V,s}$, for a vector bundle V of rank d on $X[n-s+1]$. These terms may be successively reduced using theorems 4.1.1 and 4.2.1, eventually yielding an answer in terms only of the Chow groups of X^T for $T = 1, \dots, n$. In general there is no known formula for the Chow groups of X^T in terms of the Chow groups of X , however, if X has a cellular decomposition (see definition 7.3.2), then it is true that $A(X^T) = \otimes_{a \in T} A(X)$.

5. BETTI NUMBERS AND POINCARÉ POLYNOMIALS

In this section, we analyze generating functions for the Betti numbers and Poincaré polynomials of the varieties $T_{d,n}$ in the case when $k = \mathbb{C}$. By Corollary 7.3.4, the Betti numbers coincide with the ranks of the Chow groups, and therefore, since the presentation of the Chow groups of $T_{d,n}$ is independent of the underlying field k , these determine the Chow groups in general. A recursive description of these polynomials was given in [FM94] for the spaces $X[n]$. In [Man95], Manin relates the Poincaré polynomials of these spaces as well as the polynomials for $\overline{M}_{0,n}$ to solutions to certain differential and functional equations. We apply Manin's ideas here to obtain similar results for $T_{d,n}$, which specialize to Manin's original result for $\overline{M}_{0,n+1}$ in the case $d = 1$.

Indeed, the defining equations for the generating functions of $T_{d,n}$ described here are identical to those discussed by Manin (Theorem 0.4.1) in his analysis of the Poincaré polynomials of $X[n]$. In our case, we recover these equations from the explicit blowup construction of

$T_{d,n}$, just as one can recover Theorem 0.3.1 of Manin from the blowup construction of $\overline{M}_{0,n}$ of Keel.

For a smooth compact variety Z , denote its Poincaré polynomial by $P_Z(q) = \sum_j \dim H^j(Z)q^j$. In particular, put

$$\kappa_m = P_{\mathbb{P}^{m-1}}(q) = \frac{q^{2m} - 1}{q^2 - 1}.$$

Fix d , and for $n \geq 2$, denote by $P_n(q) = P_{T_{d,n}}(q)$ the Poincaré polynomial of $T_{d,n}$. From corollary 7.3.4 and the inductive presentation of the Chow groups of $T_{d,n}$ in section 4.1, we have the following recursion for the Poincaré polynomials $P_n(q)$.

$$(1) \quad P_{n+1}(q) = (\kappa_{d+1} + nq^2\kappa_{d-1})P_n(q) + q^2\kappa_d \sum_{\substack{i+j=n+1 \\ 2 \leq i \leq n-1}} \binom{n}{i} P_i(q)P_j(q)$$

Defining $P_1(q) = 1$, and defining $p_n = p_n(q) = \frac{P_n(q)}{n!}$, this is equivalent to either of the recursions:

$$(n+1)p_{n+1} = (\kappa_{d+1} + nq^2\kappa_{d-1})p_n + q^2\kappa_d \sum_{\substack{i+j=n+1 \\ 2 \leq i \leq n-1}} jp_i p_j$$

$$(n+1)p_{n+1} = (\kappa_{d+1} + nq^2\kappa_{d-1})p_n + q^2\kappa_d \sum_{\substack{i+j=n+1 \\ i \geq 1}} jp_i p_j - q^2\kappa_d n p_n - q^2\kappa_d p_n.$$

The fact that $\kappa_{d+1} = 1 + q^2\kappa_d$, $q^2\kappa_{d-1} = \kappa_d - 1$, and that $q^2\kappa_d = q^{2d} - 1 + \kappa_d$ shows that the recursion in (1) can be rewritten as:

$$(2) \quad (n+1)p_{n+1}(q) = (1 - nq^{2d})p_n(q) + q^2\kappa_d \sum_{\substack{i+j=n+1 \\ i \geq 1}} jp_i(q)p_j(q).$$

Consider the following generating function, recalling that $p_1(q) = 1$:

$$\psi(q, t) = t + \sum_{n \geq 2} p_n(q)t^n = \sum_{n \geq 1} p_n(q)t^n.$$

Theorem 5.0.2. $\psi(q, t)$ is the unique root in $t + t^2\mathbb{Q}[q][[t]]$ of the following functional equation in t with parameter q :

$$(3) \quad \kappa_d(1 + \psi)^{q^{2d}} = q^{2d+2}\kappa_d\psi - q^{2d}(q^{2d} - 1)t + \kappa_d$$

or the following differential equation in t with parameter q :

$$(4) \quad (1 + q^{2d}t - q^2\kappa_d\psi)\psi_t = 1 + \psi.$$

Proof. First, note that we get (4) from (3) by differentiating in t . Moreover, we can see that the equations are equivalent to the recursion (1)

or (2). In particular, since $\psi_t(q, t) = \sum_{n \geq 1} np_n(q)t^{n-1}$, the t^n term for $n \geq 1$ of the left hand side of the differential equation is

$$(n+1)p_{n+1} + q^{2d}np_n - q^2\kappa_d \sum_{\substack{i+j=n+1 \\ i \geq 1, j \geq 1}} jp_jp_i$$

which is equal to p_n by (2), and for $n = 0$ is $p_1 = 1$. This is exactly the statement of the theorem. \square

Fix $d \geq 1$ and define the generating function $\eta(t) = t + \sum_{n \geq 2} \chi(T_{d,n})t^n$.

Corollary 5.0.3. *$\eta(t)$ is the unique root in $t + t^2\mathbb{Q}[q][[t]]$ of any of the following equations:*

$$\begin{aligned} d(1+\eta)\log(1+\eta) &= (d+1)\eta - t \\ (1+t-d\eta)\eta_t &= 1+\eta. \end{aligned}$$

Proof. Differentiating the first equation gives the second. The result follows since the Euler characteristic of a smooth compact variety Z can be defined by $\chi(Z) = P_z(-1)$ so that $\eta(t) = \psi(-1, t)$. \square

6. CHOW RING OF $T_{d,n}$ AND PAIRING BETWEEN DIVISORS AND CURVES

In this section we examine the structure of the Chow ring of the space $T_{d,n}$. Let k be an arbitrary field. We recall the definition of the divisors $T_{d,n}(S)$ described in Section 3. Fix $d \geq 1, n \geq 2$, and let $\delta_S = [T_{d,n}(S)]$ be the corresponding cycle class in the Chow group $A_*(T_{d,n})$. We obtain an explicit description of the Chow ring of $T_{d,n}$ by considering the Fulton-MacPherson configuration space $\mathbb{A}^d[n]$. Let $i : D(N) \hookrightarrow \mathbb{A}^d[n]$ be the inclusion of the divisor on $\mathbb{A}^d[n]$ where all the points coincide, and consider the morphism $\pi : D(N) \rightarrow \mathbb{A}^d$. By Definition 3.1.1, $T_{d,n} = \pi^{-1}(0)$. Recall that by Proposition 3.3.3, $T_{d,n} \times \mathbb{A}^d \cong D(N)$.

This implies that we may regard $D(N)$ as a (trivial) vector bundle over $T_{d,n}$, and therefore we have a morphism $\pi : D(N) \rightarrow T_{d,n}$, and a flat pullback inducing an isomorphism $(\pi^*)^{-1} : A^*D(N) \cong A^*T_{d,n}$. In particular, one may check from the definitions that if we let $D(S)$ be the divisor defined in [FM94] which we used in Section 3 then for $S \neq N$,

$$\delta_S = (\pi^*)^{-1}i^![D(S)].$$

We may similarly define $\delta_N = (\pi^*)^{-1}i^![D(N)]$. For any two distinct elements a and $b \in N$, define $\Sigma_{ab} := \sum_{S \subset N \setminus \{ab\}}$ $\delta_{\{ab\} \cup S}$.

Theorem 6.0.4. $A^*(T_{d,n}) \cong \mathbb{Z}[\{\delta_S\}_{S \subset N, 2 \leq |S| \leq n}] / I_{d,n}$, where $I_{d,n}$ is the ideal generated by:

- (1) $\delta_S \cdot \delta_T = 0$ for all $S, T \subset N$, such that $2 \leq |S|, |T|$ and $\emptyset \neq S \cap T \subsetneq S, T$;
- (2) $(\Sigma_{ij})^d = 0$, for all $i, j \in N$.

Proof. This follows from immediately from [FM94], and the isomorphism $A^*(T_{d,n}) \cong A^*(D(N))$ above. \square

In the remainder of this section, we state and prove the following pairing between 1-cycles and the boundary divisors.

Theorem 6.0.5. For $T \subsetneq N$, $|T| \geq 2$, define 1-cycles $C_T \in A_1(T_{d,n})$ by

$$C_T := \delta_T^{d(|T|-1)-1} \cdot \delta_N^{d(n-|T|)-1}.$$

If $S \subsetneq N$, $|S| \geq 2$, then

$$\delta_S \cdot C_T = \begin{cases} (-1)^{d(n-1)} & \text{if } S=T; \\ (-1)^{n-2} & \text{if } d=1, |S|=2, S \subsetneq T; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 6.0.6. For $d > 1$, the 1-cycles C_T , $|T| \geq 2$ form a \mathbb{Z} -basis for $A_1(T_{d,n})$. In the case $d=1$, the 1-cycles C_T , $|T| \geq 3$ form a \mathbb{Z} -basis for $A_1(T_{1,n}) = A_1(\overline{M}_{0,n+1})$.

Proof. First note that the set $\{\delta_T\}_{T \subset N, |T| \geq 3}$ forms a \mathbb{Z} -basis for the codimension 1-cycles on $T_{1,n}$, and that the set $\{\delta_T\}_{T \subset N, |T| \geq 2}$ forms a \mathbb{Z} -basis for the codimension 1-cycles on $T_{d,n}$.

Note that the ring described in the theorem does not depend on the choice of the base field. The statement which we have to prove is independent of the choice of k , and we may assume without loss of generality that $k = \mathbb{C}$. By Appendix 7.3.4, the space $T_{d,n}$ is an HI space. Since it is a compact smooth manifold with torsion free cohomology groups, we obtain Poincaré duality induced by the intersection pairing of divisors and curves. Therefore the C_T 's form a dual integer basis to the D_T 's, up to sign. \square

In order to prove Theorem 6.0.5, we first establish several identities. For convenience, we denote by δ_i the divisor class $\delta_{\{1,2,\dots,i\}}$. The following is an immediate consequence of Theorem 6.0.4.

Lemma 6.0.7. For $S, T \subset N$, $i \in S, T$, and $l \in T \setminus S$, we have $\delta_T \cdot \delta_S = 0$ unless $S \subset T$. In particular, if $l \in T$, then $\delta_T \cdot \delta_{l-1} = 0$ unless $\{1, \dots, l\} \subset T$.

Lemma 6.0.8. For $S \subsetneq N$, $|S| \geq 2$, $\delta_S \cdot \delta_N^{d(n-|S|)} = 0$. Consequently, if $S \neq N$, $|S| > j$, then $\delta_S \cdot \delta_N^{d(n-j)-1} = 0$.

Proof. We proceed by induction on $n - |S|$. The base case $n - |S| = 0$ holds trivially. Suppose that the result holds for $n - |S| < k$. Let $S \subset N$, $|S| = n - k$. Choose $i \in S$ and $j \notin S$. By Theorem 6.0.4, if $j \in T$ and $i \in S \cap T$, then $\delta_T \delta_S = 0$ unless $S \subsetneq T$. Moreover, for such $T \neq N$, $\delta_T \delta_N^{d(n-|S|-1)} = 0$ by the inductive hypothesis. Therefore

$$0 = (\sum_{i,j \in T} \delta_T)^d \delta_S \delta_N^{d(n-|S|-1)} = \delta_N^d \delta_S \delta_N^{d(n-|S|-1)} = \delta_S \delta_N^{d(n-|S|)}$$

since $(\sum_{i,j} \delta_{ij})^d = 0$ by Theorem 6.0.4. \square

Lemma 6.0.9. Given $2 \leq j \leq n$, if $1 \leq k < i \leq j$, $T \subset \{1, \dots, i\}$, $|T| = i - k$, then

$$\delta_T \cdot \delta_i^{kd+1} \cdot \delta_{i+1}^d \cdots \delta_j^d \cdot \delta_N^{d(n-j)-1} = 0.$$

Proof. Renumbering the elements of T , it suffices to take $\delta_T = \delta_{i-k}$. We proceed by induction on k . For the base case $k = 1$, we proceed by induction on $j - i$ with base case $i = j$. By Lemma 6.0.7, if $1, j \in T$, then $\delta_T \delta_{j-1} = 0$ unless $\{1, \dots, j\} \subset T$. Also note that by Lemma 6.0.8, if $T \neq N$ and $|T| > j$, then $\delta_T \delta_N^{d(n-j)-1} = 0$. The relation $(\sum_{1j})^d = 0$ of Theorem 6.0.4 gives

$$0 = (\sum_{1,j \in T} \delta_T)^d \cdot \delta_{j-1} \cdot \delta_j \cdot \delta_N^{d(n-j)-1} = (\delta_j + \delta_N)^d \cdot \delta_{j-1} \cdot \delta_j \cdot \delta_N^{d(n-j)-1}.$$

The summands coming from the terms of the binomial expansion of $(\delta_j + \delta_N)^d$ with positive degree in δ_N vanish by Lemma 6.0.8 for $\delta_S = \delta_j$, giving the result.

Now suppose that the result holds for integers less than $j - i$. If $1, i \in T$, then $\delta_T \delta_{i-1} = 0$ unless $\{1, \dots, i\} \subset T$ by Lemma 6.0.7. Indeed, $\delta_T \delta_{i-1} \delta_i \cdots \delta_j = 0$ unless $T = \delta_i, \dots, \delta_j$ or $\{1, \dots, j\} \subsetneq T$. The relation $(\sum_{1i})^d = 0$ of Theorem 6.0.4 gives

$$0 = (\sum_{1,i \in T} \delta_T)^d \cdot \delta_{i-1} \cdot \delta_i \cdot \delta_{i+1}^d \cdots \delta_j^d \cdot \delta_N^{d(n-j)-1}$$

We see that terms involving $|T| > j$ vanish by Lemma 6.0.8. Since terms involving $\delta_T = \delta_{i+1}, \dots, \delta_j$ vanish by the inductive hypothesis, we have established the base case $k = 1$.

Suppose that the result holds for integers less than k , and consider the following identity:

$$0 = (\sum_{1,i-k+1 \in T} \delta_T)^d \cdot \delta_{i-k} \cdot \delta_i^{(k-1)d+1} \cdot \delta_{i+1}^d \cdots \delta_j^d \cdot \delta_N^{d(n-j)-1}$$

By Lemma 6.0.7, $\delta_T \delta_i \cdots \delta_j = 0$ unless $\{1, \dots, i - k + 1\} \subset T \subset \{1, \dots, i\}$, $\{1, \dots, j\} \subset T$, or $\delta_T = \delta_i, \dots, \delta_j$. Terms involving δ_T ,

of the first type vanish by the inductive hypothesis, and terms involving $|T| > j$ are zero by Lemma 6.0.8. Therefore, nonzero terms can involve only $\delta_T = \delta_i, \dots, \delta_j$. Finally, the terms with contributions from $\delta_{i+1}, \dots, \delta_j$ vanish by the base case $k = 1$. Hence we are left with

$$0 = \delta_i^d \delta_{i-k} \cdot \delta_i^{(k-1)d+1} \cdot \delta_{i+1}^d \cdots \delta_j^d \cdot \delta_N^{d(n-j)-1},$$

proving the proposition. \square

Lemma 6.0.10. For $j \geq 1$, $\delta_2^d \cdots \delta_j^d \cdot \delta_N^{d(n-j)-1} = (-1)^{j-1} \delta_N^{d(n-1)-1}$.

Proof. We prove the result by induction on j . The result holds trivially for $j = 1$. For the base case $j = 2$, it follows from Lemma 6.0.8 that

$$0 = (\sum_{1,2 \in T} \delta_T)^d \cdot \delta_N^{d(n-2)-1} = \delta_2^d \cdot \delta_N^{d(n-2)-1} + \delta_N^d \delta_N^{d(n-2)-1}.$$

Suppose that the result holds for integers less than j , and consider

$$0 = (\sum_{1,j \in T} \delta_T)^d \cdot \delta_2^d \cdots \delta_{j-1}^d \cdot \delta_N^{d(n-j)-1}.$$

If $1, j \in T$, then $\delta_T \delta_{j-1} = 0$ unless $\{1, \dots, j\} \subset T$ by Lemma 6.0.7. Nonzero terms in the expansion can only involve $\delta_T = \delta_j, \delta_N$ by Lemma 6.0.8; moreover, terms involving both δ_j and δ_N vanish. Therefore

$$\delta_2^d \cdots \delta_j^d \cdot \delta_N^{d(n-j)-1} = -\delta_2^d \cdots \delta_{j-1}^d \cdot \delta_N^{d(n-(j-1))-1} = -(-1)^{j-2} \delta_N^{n-2}$$

as needed, where the final equality holds by the inductive hypothesis. \square

Proposition 6.0.11. For $1 \leq j \leq n$, if $1 \leq i \leq j$ and $1 \leq k \leq j - i$, then

$$\delta_2^d \cdots \delta_i^d \cdot \delta_{i+k}^{dk} \cdot \delta_{i+k+1}^d \cdots \delta_j^d \cdot \delta_N^{d(n-j)-1} = (-1)^{j-k} \delta_N^{d(n-1)-1}.$$

Proof. We prove the result by induction on $j - i$. The base case $j - i = 1$ so that $k = 1$ is Lemma 6.0.10. Suppose that the statement holds for integers less than $j - i$. Consider the identity:

$$0 = (\sum_{1, i+1 \in T} \delta_T)^d \cdot \delta_i \cdot \delta_{i+k}^{d(k-1)} \cdot \delta_{i+k+1}^d \cdots \delta_j^d \cdot \delta_N^{d(n-j)-1}$$

Lemma 6.0.7 implies that nonzero contributions involve only δ_T with $\{1, \dots, i+1\} \subset T$. Terms involving δ_T with $|T| > j$ vanish by Lemma 6.0.8, those involving δ_T with $|T| < i+k$ vanish by Lemma 6.0.9, as do those involving $\delta_{i+k+1}, \dots, \delta_j$. Terms involving both δ_{i+1} and δ_{i+k} vanish by Lemma 6.0.9. Hence

$$0 = \delta_i \cdot \delta_{i+1}^d \cdot \delta_{(i+1)+(k-1)}^{d(k-1)} \cdot \delta_{i+k+1}^d \cdots \delta_j^{d(n-j)-1} + \delta_i \cdot \delta_{i+k}^{d(k)} \cdot \delta_{i+k+1}^d \cdots \delta_j \delta_N^{d(n-j)-1}.$$

Multiplying by $\delta_2^d \cdots \delta_{i-1}^d \delta_i^{d-1}$ and applying the inductive hypothesis gives the result:

$$\begin{aligned} & \delta_2^d \cdots \delta_i^d \cdot \delta_{i+k}^{d(k)} \cdot \delta_{i+k+1}^d \cdots \delta_j^d \cdot \delta_N^{d(n-j)-1} \\ &= -\delta_2^d \cdots \delta_{i+1}^d \cdot \delta_{(i+1)+(k-1)}^{d(k-1)} \cdot \delta_{i+k+1}^d \cdots \delta_j^d \cdot \delta_N^{d(n-j)-1} = -(-1)^{j-(k-1)} \delta_N^{d(n-1)-1}. \end{aligned}$$

□

Proposition 6.0.12. $\delta_N^{d(n-1)-1} = (-1)^{d(n-1)-1}$.

Before proving this proposition, we give a proof of Theorem 6.0.5.

Proof of Theorem 6.0.5. Let $T \subsetneq N$ with $|T| \geq 2$. Without loss of generality, say $T = \{1, \dots, j\}$. Then $\delta_T \cdot C_T$ is equal to

$$\delta_j^{d(j-1)} \cdot \delta_N^{d(n-j)-1} = -\delta_N^{d(n-1)-1} = (-1)^{d(n-1)}$$

by Proposition 6.0.11 for $i = 1$, $k = j - 1$ and Proposition 6.0.12.

Let $S \subsetneq N$. If $\emptyset \neq S \cap T \subsetneq S, T$, then $\delta_S \cdot C_T = 0$ by Theorem 6.0.4. If $S \cap T = \emptyset$, $s \in S$, then

$$0 = (\sum_{1, s \in T'} \delta_{T'})^d \cdot \delta_S \cdot \delta_j \cdot \delta_N^{d(n-j-1)-1} = \delta_S \cdot \delta_j \cdot \delta_N^{d(n-j)-1}$$

since nonzero summands only involve $\delta_{T'}$ with $T \cup \{s\} \subset T'$ by Lemma 6.0.7, and all these contribute zero unless $T' = N$ by Lemma 6.0.8. Therefore $\delta_S \cdot C_T = 0$. If $T \subsetneq S$, then $\delta_S \cdot C_T = 0$ by Lemma 6.0.8.

Suppose $S \subsetneq T$. We may assume that $S = \{1, \dots, j-k\}$ with $k \geq 1$. Then

$$\delta_S \cdot C_T = \delta_{j-k} \cdot \delta_j^{d(j-1)-1} \cdot \delta_N^{d(n-j)-1}.$$

It follows from Lemma 6.0.9 that this is zero when $d(j-1)-1 \geq dk+1$, or equivalently when $d(j-k-1) \geq 2$. We conclude that $\delta_S \cdot C_T = 0$ if $d = 1$ and $|S| \geq 3$, or if $d \geq 2$. If $d = 1$ and $|S| = 2$, then

$$\delta_S \cdot C_T = \delta_2 \delta_j^{j-2} \delta_N^{(n-j)-1} = \delta_N^{n-2} = (-1)^{n-2}.$$

by Proposition 6.0.11 for $i = 2$, $k = j - 2$ and Proposition 6.0.12. □

It remains to prove Proposition 6.0.12. Since the proof involves spaces $T_{d,n}$ for varying n , for the remainder of this section, we use the more precise notation $\delta_{S,n} = [T_{d,n}(S)]$. In this language, we need to show

$$\int_{T_{d,n}} \delta_{N,n}^{d(n-1)-1} = (-1)^{d(n-1)-1}.$$

We first establish the following lemmas.

Lemma 6.0.13.

$$\int_{T_{1,3}} [T_{1,3}(\{1, 2, 3\})] = -1 \text{ and } \int_{T_{d,2}} [T_{d,2}(\{1, 2\})]^{d-1} = (-1)^{d-1}.$$

Proof. From Lemma 3.4.1, we know that $T_{d,2} \cong \mathbb{P}^{d-1}$. As discussed in the beginning of the section, $[T_{d,2}(\{1, 2\})]$ corresponds to $i^*[D(\{1, 2\})]$, where i is the inclusion $D(\{1, 2\}) \hookrightarrow A^d[2]$. By [Ful98], we have

$$i^*[D(\{1, 2\})] = c_1(N_{D(\{1,2\})\mathbb{A}^d[2]}) = \mathcal{O}_{D(\{1,2\})}(-1).$$

Hence $[T_{d,2}(\{1, 2\})] = c_1(\mathcal{O}_{\mathbb{P}^{d-1}}(-1))$, and the result follows.

Write $T_{1,3} \cong \mathbb{P}^1$. As above, for i the inclusion $D(\{1, 2, 3\}) \hookrightarrow \mathbb{A}^1[3]$, $i^*[D(\{1, 2, 3\})]$ corresponds to $\mathcal{O}_{\mathbb{P}^1}(-1)$ on $T_{1,3}$, which gives the result. \square

Let $\pi_{n+1} : T_{d,n+1} \rightarrow T_{d,n}$ be the map which drops the $(n+1)$ st marking as described in Section 3.

Lemma 6.0.14. $\pi_{n+1}^*(\delta_{N,n}) = \delta_{N,n+1} + \delta_{N^+,n+1}$.

Proof. As in the beginning of the section, let $i : D_{\mathbb{A}^d[n]}(N) \hookrightarrow \mathbb{A}^d[n]$ be the inclusion, and $p : D_{\mathbb{A}^d[n]}(N) \rightarrow T_{d,n}$ the vector bundle morphism. Let $\pi = \pi_{n+1}$ and consider the commutative diagram given by dropping the $n+1$ 'st point:

$$\begin{array}{ccccc} T_{d,n+1} & \xleftarrow{p_+} & D_{\mathbb{A}^d[n+1]}(N^+) & \xrightarrow{i_+} & X[n+1] \\ \downarrow \pi & & \downarrow \pi' & & \downarrow \pi'' \\ T_{d,n} & \xleftarrow{p} & D_{\mathbb{A}^d[n]}(N) & \xrightarrow{i} & X[n], \end{array}$$

where we have used the subscripts to distinguish which space the divisor sits on. By [FM94], Proposition 3.4, $(\pi'')^*[D_{\mathbb{A}^d[n]}(N)] = [D_{\mathbb{A}^d[n+1]}(N^+)] + [D_{\mathbb{A}^d[n+1]}(N)]$. Therefore, by commutativity of the diagram, we have :

$$\begin{aligned} \pi^*(\delta_{N,n}) &= \pi^*(p_+^*)^{-1} i_+^*[D_{\mathbb{A}^d[n]}(N)] = (p_+^*)^{-1} i_+^*(\pi'')^*[D_{\mathbb{A}^d[n]}(N)] \\ &= (p_+^*)^{-1} i_+^*([D_{\mathbb{A}^d[n+1]}(N^+)] + [D_{\mathbb{A}^d[n+1]}(N)]) = \delta_{N,n+1} + \delta_{N^+,n+1}. \end{aligned}$$

\square

Lemma 6.0.15. *Let $\pi = \pi_{n+1} : T_{d,n+1} \rightarrow T_{d,n}$ be the map which drops the $(n+1)$ st marking as described in Section 3. Then*

$$\int_{T_{d,n+1}} \delta_{N^+,n+1}^{dn-1} = (-1)^d \int_{T_{d,n}} \pi_* \left(\delta_{N,n+1}^d \right) \cdot \delta_{N,n}^{d(n-1)-1}.$$

Proof. We first note that, solving for $\delta_{N^+,n+1}$ in Lemma 6.0.14 gives

$$\begin{aligned} \pi_*(\delta_{N^+,n+1}^d) &= (\pi^* \delta_{N,n} - \delta_{N,n+1})^d \\ &= \sum_{i=0}^d \binom{d}{i} (-1)^i \pi_*(\delta_{N,n}^i) \cdot \delta_{N,n+1}^{d-i} = (-1)^d \pi_*(\delta_{N,n+1}^d), \end{aligned}$$

since $\dim T_{d,n} = d(n-1)$, so that $\pi_*(\delta_{N,n+1}^i) = 0$ for $i < d$.

Again solving for $\delta_{N^+,n+1}$ in Lemma 6.0.14, we have

$$\delta_{N^+,n+1}^{dn-1} = \delta_{N^+,n+1}^d \cdot \left(\pi^* \delta_{N,n} - \delta_{N,n+1} \right)^{d(n-1)-1}.$$

Since $\delta_{N^+,n+1}^d \cdot \delta_{N,n+1} = 0$ by Lemma 6.0.8, the summands from the binomial expansion vanish for any positive power of $\delta_{N,n+1}$. Hence

$$\delta_{N^+,n+1}^{dn-1} = \delta_{N^+,n+1}^d \cdot (\pi^* \delta_{N,n})^{d(n-1)-1} = \delta_{N^+,n+1}^d \cdot \pi^*(\delta_{N,n}^{d(n-1)-1}).$$

The projection formula and the fact above gives the result. \square

For the next two lemmas, we follow the notation of Section 3.

Lemma 6.0.16. *Writing $F_{d,n}^1 = \mathbb{P}_{T_{d,n}}(\mathcal{L}_N \oplus V_{T_{d,n}})$, we may identify $F_{d,n}^1(N) \subset F_{d,n}^1$ with the subbundle $\mathbb{P}_{T_{d,n}}(V_{T_{d,n}})$.*

Proof. We recall that the functor defining $F_{d,n}^1$ takes a variety H to the collection of compatible screens $(i_S : \mathcal{L}_S \rightarrow (V_H)^S/V_H)$ where either $S \subset N$ or $S = N^+$. Equivalently, examining the proof of Theorem 3.1.2, the data of i_{N^+} amounts to choosing a vector bundle inclusion $j : \mathcal{L}_{N^+} \rightarrow \mathcal{L}_N \oplus V_H$. The morphism i_{N^+} is then obtained by using i_N to map \mathcal{L}_N into $(V_H)^N/V$ and identifying $(V_H)^{N^+}/V_H \cong (V_H)^N/V_H \oplus V_H$.

The subfunctor represented by $F_{d,n}^1(N)$ is defined by requiring that all the compatibility morphisms $\mathcal{L}_{N^+} \rightarrow \mathcal{L}_S$ are zero for $S \subset N$. By compatibility, it suffices to know that $\mathcal{L}_{N^+} \rightarrow \mathcal{L}_N$ is zero, or that the morphism j maps \mathcal{L}_{N^+} entirely inside of $V_H \subset \mathcal{L}_{N^+} \oplus V_H$. But this just says that $F_{d,n}^1(N) = \mathbb{P}_{T_{d,n}}(V_{T_{d,n}}) \subset \mathbb{P}_{T_{d,n}}(\mathcal{L}_N \oplus V_{T_{d,n}})$, as desired. \square

Lemma 6.0.17. *Let $\rho : T_{d,n+1} \rightarrow F_{d,n}^1$ be the natural projection. Then $\rho^*([F_{d,n}^1(N)]) = [T_{d,n+1}(N)]$.*

Proof. We first note that scheme-theoretically $\rho^{-1}(F_{d,n}^1(N)) = T_{d,n+1}(N)$. To see this, we note that a morphism from a scheme H to $\rho^{-1}(F_{d,n}^1(N))$ is given by specifying a collection of compatible screens $(\mathcal{L}_S \rightarrow (V_H)^S/V_H)$ for $S \subset N^+$ such that the compatibility morphism $\mathcal{L}_{N^+} \rightarrow \mathcal{L}_N$ is zero. But it is easy to check that this is precisely equivalent to the morphism being in $T_{d,n+1}(N)$.

Now the pullback $\rho^*([F_{d,n}^1(N)])$ is represented by a class on the scheme theoretic inverse image. Since ρ is of relative dimension zero, the pullback is represented by a multiple of the fundamental class of the inverse image $[T_{d,n+1}(N)]$. But since $\rho_*\rho^* = id$, this multiple must be 1. \square

Lemma 6.0.18. *With the notation of Lemma 6.0.15,*

$$\pi_*\left(\delta_{N,n+1}^d\right) = [T_{d,n}].$$

Proof of Lemma 6.0.18. Consider the commutative diagram of natural morphisms

$$\begin{array}{ccc} T_{d,n+1} & & \\ \beta \downarrow & \searrow \pi & \\ F_{d,n}^1 & \xrightarrow{\rho} & F_{d,n}^0 = T_{d,n}. \end{array}$$

By Lemma 6.0.17, $[T_{d,n+1}(N)] = \beta^*[F_{d,n}^1(N)]$, and so

$$\begin{aligned} \pi_*([T_{d,n+1}(N)]^d) &= \pi_*\beta^*([F_{d,n}^1(N)]^d) = \rho_*\beta_*\beta^*([F_{d,n}^1(N)]^d) \\ &= \rho_*\beta_*\beta^*([F_{d,n}^1(N)]^d) = \rho_*([F_{d,n}^1(N)]^d) \end{aligned}$$

By Lemmas 6.0.16 and 7.1.1, we have $\rho_*([F_{d,n}^1(N)]^d) = [T_{d,n}]$, completing the proof. \square

We now use these lemmas to prove the main result of this section, Proposition 6.0.12.

Proof of Proposition 6.0.12. We proceed by induction on n . The base cases $d = 1, n = 3$, and $d > 1, n = 2$ are proved in Lemma 6.0.13. Then

$$\int_{T_{d,n+1}} \delta_{N^+,n+1}^{dn-1} = (-1)^d \int_{T_{d,n}} \delta_{N,n}^{d(n-1)-1} = (-1)^d (-1)^{dn-1} = (-1)^{d(n+1)-1},$$

as needed, where the first equality follows from Lemma 6.0.15 and Lemma 6.0.18, and the second equality follows from the inductive hypothesis. \square

6.1. Conjectural Pairing of cycles. Let \mathcal{S} be a collection of non-overlapping subsets of N . For subsets $S, T \in \mathcal{S}$, we use the notation $S \prec T$ to mean that $S \subset T$ and for every $U \in \mathcal{S}$ such that $U \subset T$, we have $U \subset S$. Previously we have been calling this relation “ S is a child of T .”

Definition 6.1.1. We define the following symbols:

- (1) For $S \in \mathcal{S}$, $ch(S) = \{T \mid T \prec S\}$.
- (2) For $S \in \mathcal{S}$, $\chi(S) = |S| - \sum_{T \in ch(S)} |T| + |ch(S)| - 1$.

The conjectural formula is as follows:

$$\left(\prod_{N \neq S \in \mathcal{S}} \delta_S^{d\chi(S)} \right) \delta_N^{d\chi(N)-1} = \pm \delta_N^{dn-d-1}.$$

This gives the following conjectural pairing: For the cycle $\prod_{S \in \mathcal{S}} \delta_S^{n_S}$, where each $n_i > 0$, we conjecture that

$$\left(\prod_{S \in \mathcal{S}} \delta_S^{n_S} \right) \left(\prod_{N \neq S \in \mathcal{S}} \delta_S^{d\chi(S)-n_S} \right) \delta_N^{d\chi(N)-n_N-1} = \pm \delta_N^{dn-d-1}.$$

In other words, $\prod_{S \in \mathcal{S}} \delta_S^{n_S}$ is “dual” to the cycle

$$\left(\prod_{N \neq S \in \mathcal{S}} \delta_S^{d\chi(S)-n_S} \right) \delta_N^{d\chi(N)-n_N-1}.$$

Example 6.1.2. For $\mathcal{S} = \{S\}$, the codimension 1-cycle δ_S is dual to $\delta_S^{d(|S|-0+0-1)} \delta_N^{d(n-s+1-1)-1}$.

Example 6.1.3. For $\mathcal{S} = \{S, T\}$ with $S \subsetneq T$, the codimension 2-cycle $\delta_S \delta_T$ is dual to $\delta_S^{d(|S|-0+0-1)} \delta_T^{d(|T|-|S|+1-1)} \delta_N^{d(|N|-|T|+1-1)-1}$.

Example 6.1.4. For $\mathcal{S} = \{S, T\}$ with $S \cap T = \emptyset$, $\delta_S \delta_T$ is dual to $\delta_S^{d(|S|-0+0-1)} \delta_T^{d(|T|-0+0-1)} \delta_N^{d(|N|-|S|-|T|+2-1)-1}$.

7. APPENDIX

7.1. An intersection-theoretic Lemma.

Lemma 7.1.1. *Let X be a variety, and suppose $\pi : E \rightarrow X$ is a vector bundle of rank $d+1$ and $V \subset E$ is a rank d subvector bundle. Consider the inclusion $i : P(V) \hookrightarrow P(E)$. Then $\pi_*((i_*[P(V)])^d) = [X]$ in the Chow group of X .*

In the proof of this fact we omit the i_* for notational convenience.

Proof. Let us first consider the case where $E = L \oplus V$ for some trivial line bundle L . By [Ful98], Lemma 3.3, $c_1(\mathcal{O}_E(1)) \cap [P(E)] = [P(V)]$. Using [Ful98], Proposition 3.1, we have $[X] = s_0(E) \cap [X]$. But by the definition of the Segre class, we have:

$$\begin{aligned} [X] &= s_0(E) \cap [X] = \pi_*(c_1(\mathcal{O}_E(1))^d \cap [P(E)]) \\ &= \pi_*([P(V)]^d \cap [P(E)]) = \pi_*([P(V)]^d) \end{aligned}$$

Now let us consider the general case. By counting dimensions, we know that $\pi_*([P(V)]^d) \in A_{\dim(X)}(X) = \mathbb{Z}[X]$, and so $\pi_*([P(V)]^d) = a[X]$ for some $a \in \mathbb{Z}$. We show that $a = 1$. Choose an open subvariety $j : U \hookrightarrow X$ such that $E|_U = L|_U \oplus V|_U$ for some trivial line bundle L .

We have a commutative diagram:

$$\begin{array}{ccccc}
P(V|_U) & \xrightarrow{i_U} & P(E|_U) & \xrightarrow{\pi_U} & U \\
j_V \downarrow & & j_E \downarrow & & j \downarrow \\
P(V) & \xrightarrow{i} & P(E) & \xrightarrow{\pi} & X
\end{array}$$

Now we observe that $j^*\pi_*([P(V)]^d) = j^*(a[X]) = a[U]$. But by [Ful98] Propositions 1.7 and 2.3(d), this gives:

$$\begin{aligned}
a[U] &= j^*\pi_*([P(V)]^d) = (\pi_U)_*j_E^*([P(V)]^d) \\
&= (\pi_U)_*((j_E^*[P(V)])^d) = (\pi_U)_*([P(V|_U)]^d)
\end{aligned}$$

But by the first case, we have $(\pi_U)_*([P(V|_U)]^d) = [U]$, which implies that $a = 1$. \square

7.2. Inverse image of a height 1 prime ideal sheaf.

Lemma 7.2.1. *Suppose $f : X \rightarrow Y$ is a dominant morphism of varieties and $\mathcal{I} \subset \mathcal{O}_Y$ is the ideal sheaf of a codimension 1 subvariety $Z \subset Y$. If Y is locally factorial then the canonical map of coherent sheaves $f^*(\mathcal{I}) \rightarrow f^{-1}(\mathcal{I})$ is an isomorphism.*

Proof. The result is local, and so we may assume that $X = \text{Spec}(S)$, $Y = \text{Spec}(R)$, and $\mathcal{I} = \tilde{I}$ for an ideal $I \subset R$. By the hypotheses we may also assume that R and S are domains, ϕ is injective, and that R is factorial, and therefore by [Eis95], Corollary 10.6, $I = aR$ for some $a \in R$. Let $\phi : R \rightarrow S$ be the map induced by f . Then we need to show that $I \otimes_R S \cong \phi(I)S$, where the isomorphism is induced by the multiplication map. Since $I = aR$, this amounts to showing that the morphism $S \rightarrow S$ induced by multiplication by $\phi(a)$ is injective. But this is true since ϕ is injective and S is a domain. \square

7.3. HI Spaces. In this section we show that if the ground field is the complex numbers, then the Chow groups of $T_{d,n}$ are isomorphic to the homology (and cohomology) groups.

Definition 7.3.1. A complex algebraic variety X is an *HI space* if the canonical map $cl_X : A_i(X) \rightarrow H_{2i}(X)$ is an isomorphism.

Definition 7.3.2. A variety X is *cellular* if it has a filtration $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$, such that $X_i \setminus X_{i-1}$ is a disjoint union of affine spaces \mathbb{A}^i .

If X has a cellular decomposition then it is an HI space. This fact can be found in [Ful98], example 19.1.11(b). Also, if $Y \subset X$ is a closed subscheme, $U = X \setminus Y$ the open complement, and cl_Y , and cl_U are both isomorphisms, then so is cl_X . That is, Y, U both HI implies X HI. This is in [Ful98], example 19.1.11(a). It is also easy to check that if X is a projective bundle over Y , and Y is an HI space, then so is X .

We recall the following fact:

Theorem 7.3.3 ([Kee92]). *Suppose Y is a closed subvariety of X , and X and Y are both HI. Then The blowup of X along Y is also HI.*

From this, our description of the space $T_{d,n}$ as a blowup of a projective bundle shows that:

Corollary 7.3.4. *$T_{d,n}$ is an HI space.*

We similarly have:

Corollary 7.3.5. *Suppose that if X is an HI space. Then so is the Fulton-MacPherson configuration space $X[n]$.*

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