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Winning Paths In N-By-Infinity Hex

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Abstract

In $n \times \infty$ Hex the player whose task is to complete a finite path can win, but the other player can ensure that any winning path contains at least $n + \lfloor \frac{n-2}{4} \rfloor$ cells.

1. Introduction

Suppose that Blue and Red are playing Hex on an $n \times \infty$ board, with Blue being the player whose task is to complete a finite path. Blue can win, whoever plays first; this follows from the fact that Blue can win playing first or second on an $n \times (n + 1)$ board, which is proved in [2] and [3]. But how long must the winning path be? We show that Red can force the winning path to contain at least $n + \lfloor \frac{n-2}{4} \rfloor$ cells.

This partially answers a question raised by Garikai Campbell in this Journal [1]. Campbell defines the game Hex(n,ℓ) to be the same as $n \times n$ Hex, but with the additional rule that in order to win, the first player must complete a winning path consisting of at most ℓ cells. He then defines

$$\lambda(n) = \min\{\ell \mid \text{player one has a winning strategy for Hex}(n,\ell)\}.$$

A routine case analysis shows that $\lambda(n) = n$ for $n = 1, 2, 3$. Campbell shows that $\lambda(4) = 5$ and that $\lambda(n) \geq n + 1$ for $n \geq 4$. Our present result implies that $\lambda(n) \geq n + \lfloor \frac{n-2}{4} \rfloor$.

Both [1] and [3] discuss the rules and history of Hex and provide extensive references.
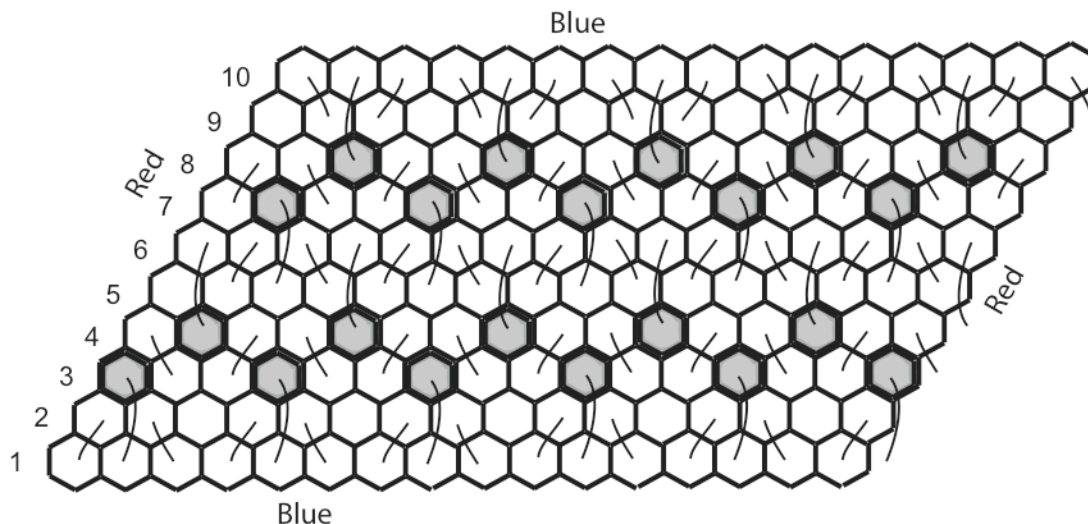
Figure 1: Red's strategy

2. Theorem and Proof

For generality consider $n \times m$ Hex. We arrange the board so that it consists of n horizontal rows. Blue tries to connect the top and bottom, and Red tries to prevent the connection or at least force the winning path to be as long as possible.

Theorem   *In $n \times m$ Hex with Blue having the two sides with m cells, whichever player moves first, Red can ensure that any Blue winning path has at least $n + \lfloor \frac{n-2}{4} \rfloor$ cells.*

The result holds for any positive integers m and n, but is nontrivial only when $6 \leq n \leq m$.

Proof.   Figure 1 shows Red's strategy in the case of a 10×16 board. Most of the cells are partitioned into pairs, and Red's strategy is to play in at least one cell of each pair—if Blue takes one, Red takes the other. To see that this strategy has the desired effect, note the following:

1. Any Blue path through rows 3 and 4 includes one of the shaded cells in those rows.

2. Blue cannot connect row 1 to a shaded cell in row 3 with a path of just three cells. Therefore any winning Blue path that contains a shaded cell in row 3 must contain at least two cells in row 3 or at least two cells in row 2.

3. Similarly, any winning Blue path containing a shaded cell in row 4 must contain at least two cells in row 4 or at least two cells in row 5. Therefore any winning Blue path must contain at least 5 cells in the four rows 2, 3, 4, 5.

4. Similarly, any winning Blue path must contain at least 5 cells in the four rows 6, 7, 8, 9.

In general, a winning path needs five cells to cross each block of four rows, excluding the first and last rows. If there are n rows, a winning path must include $n + \lfloor \frac{n-2}{4} \rfloor$ cells. $\square$

Open question: Can Blue win on a $5 \times \infty$ board with a path of length 5?

Acknowledgments

References

1. Garikai Campbell, On optimal play in the game of Hex, *INTEGERS: Electronic Journal of Combinatorial Number Theory* **4** (2004), #G02.

2. Martin Gardner, *The Scientific American Book of Mathematical Puzzles and Diversions*, Simon and Schuster, New York, 1959, chapter titled "The Game of Hex," 73–83

3. Ryan B. Hayward and Jack van Rijswijck, Hex and Combinatorics, *Discrete Math* **306** (2006), 2515–2528.