

Swarthmore College

## Works

---

Mathematics & Statistics Faculty Works

Mathematics & Statistics

---

3-1-2000

# Isometric Extensions Of Zero Entropy $Z^d$ Loosely Bernoulli Transformations

Aimee S.A. Johnson  
Swarthmore College, [aimee@swarthmore.edu](mailto:aimee@swarthmore.edu)

A. A. Şahin

Follow this and additional works at: <https://works.swarthmore.edu/fac-math-stat>



Part of the [Mathematics Commons](#)

[Let us know how access to these works benefits you](#)

---

### Recommended Citation

Aimee S.A. Johnson and A. A. Şahin. (2000). "Isometric Extensions Of Zero Entropy  $Z^d$  Loosely Bernoulli Transformations". *Transactions Of The American Mathematical Society*. Volume 352, Issue 3. 1329-1343. <https://works.swarthmore.edu/fac-math-stat/6>

This work is brought to you for free by Swarthmore College Libraries' Works. It has been accepted for inclusion in Mathematics & Statistics Faculty Works by an authorized administrator of Works. For more information, please contact [myworks@swarthmore.edu](mailto:myworks@swarthmore.edu).

## ISOMETRIC EXTENSIONS OF ZERO ENTROPY $\mathbb{Z}^d$ LOOSELY BERNOULLI TRANSFORMATIONS

AIMEE S. A. JOHNSON AND AYŞE A. ŞAHİN

ABSTRACT. In this paper we discuss loosely Bernoulli for  $\mathbb{Z}^d$  actions. In particular, we prove that extensions of zero entropy, ergodic, loosely Bernoulli  $\mathbb{Z}^d$  actions are also loosely Bernoulli.

### 1. INTRODUCTION AND SUMMARY OF RESULTS

In one dimension, a zero entropy transformation is loosely Bernoulli (LB) if there is one name up to the  $\bar{f}$  metric. Intuitively, in one dimension this metric measures the proportion of indices between two names which can be matched in an order preserving way. In higher dimensions, the  $\bar{f}$  metric measures how the relative configuration of the indices in the  $d$ -dimensional names are related. It is nontrivial to extend results to higher dimensions, due to the more complicated geometry of the definition of LB.

Many of the basic properties of one dimensional LB transformations are established in [5]. In [2] we proved that certain rank 1  $\mathbb{Z}^d$  actions are LB. The arguments used the inherent geometry of the orbits of these actions. In this paper we develop a more general “nesting” machinery than was needed in [2].

The paper is organized as follows. In section 2 we remind the reader of the higher dimensional definition of  $\bar{f}$ . We also define a new matching condition and discuss its relationship to the  $\bar{f}$  metric. In particular, the results in this section provide some insight into the geometry of an  $\bar{f}$ -small permutation.

In section 3 we define properties of processes which appear weaker than LB, and prove that they are in fact equivalent to LB. This section contains the core of the new higher dimensional machinery.

In the final two sections we use our machinery to prove that  $k$  point and isometric extensions of ergodic, measure preserving and zero entropy LB  $\mathbb{Z}^d$  actions are LB. Some of the arguments we provide are parallel to those in [5], but we include them to show that they can be carried through with the higher dimensional machinery. We do not assume any familiarity on the part of the reader with the results in [5].

---

Received by the editors November 13, 1997.

1991 *Mathematics Subject Classification.* Primary 28D15; Secondary 28D20.

*Key words and phrases.* Ergodic theory,  $\mathbb{Z}^d$  actions, loosely Bernoulli actions, group extensions.

The research of the first author was partially supported by the Swarthmore College Research Fund.

The research of the second author was partially supported by the NSF under grant number DMS-9501103, and NDEPSCoR under grant number OSR-9452892.

2. THE  $\bar{f}$ -METRIC AND MATCHING NAMES

Let  $(X, \mathcal{A}, \mu)$  be a Lebesgue probability space. Take  $T$  to be an ergodic, zero entropy  $\mathbb{Z}^d$  action on  $(X, \mathcal{A}, \mu)$ . We can think of  $T$  as being generated by  $d$  commuting measure preserving one dimensional transformations on  $X$ ,  $\{T_{\vec{e}_1}, \dots, T_{\vec{e}_d}\}$ , where the set  $\{\vec{e}_1, \dots, \vec{e}_d\}$  is the standard basis for  $\mathbb{Z}^d$ . Then  $T_{\vec{v}}(x) = T_{\vec{e}_1}^{v_1} \circ \dots \circ T_{\vec{e}_d}^{v_d}(x)$ , where  $\vec{v} = (v_1, \dots, v_d)$ . We call  $(X, \mathcal{A}, \mu), T$  a  $\mathbb{Z}^d$ -dynamical system. Often we will simply write  $(X, T)$ .

For  $n \in \mathbb{N}$ , let  $B_n = \{\vec{v} \in \mathbb{Z}^d : 0 \leq v_i < n, 1 \leq i \leq d\}$ . We define the  $\epsilon$ -interior of  $B_n$  to be the collection of indices in  $B_n$  which are at least a distance  $\epsilon n$  from the boundary of  $B_n$ . For a vector  $\vec{v} \in \mathbb{Z}^d$ , we set  $\|\vec{v}\| = \max\{|v_i| : 1 \leq i \leq d\}$ .

For  $n_1 < n$  we define partitions of  $B_n$  into  $n_1$ -grids. For the first grid, imagine horizontal and vertical lines in  $B_n$  drawn at the multiples of  $n_1$ , starting at  $\vec{0}$ . More formally, let

$$r_{\vec{0}} = \{(k_1 n_1, k_2 n_1, \dots, k_d n_1) : 0 \leq k_i \leq \left\lfloor \frac{n}{n_1} \right\rfloor, i = 1, \dots, d\}$$

and set  $R_{\vec{0}} = \{B_{n_1} + \vec{u} : \vec{u} \in r_{\vec{0}}\} \cap B_n$ . We will call  $R_{\vec{0}}$  an  $n_1$ -grid of  $B_n$ , the translates of  $B_{n_1}$  will be called the grid boxes, and the vectors  $\vec{u} \in r_{\vec{0}}$  will be called the base points of the grid. We obtain all the  $n_1$ -grids of  $B_n$  by translating the grid  $R_{\vec{0}}$  by all vectors  $\vec{v} \in B_{n_1}$ . We set  $R_{\vec{v}} = (R_{\vec{0}} + \vec{v}) \cap B_n$  and note that  $r_{\vec{v}} = r_{\vec{0}} + \vec{v}$  is the set of base points of the grid  $R_{\vec{v}}$ . If  $C \subset B_n$ , we say  $R_{\vec{v}} \cap C$  is an  $n_1$ -grid of  $C$ , for any  $\vec{v} \in B_{n_1}$ .

Let  $P$  be a measurable, finite partition on  $X$  with label set  $\{p_1, \dots, p_h\}$ .  $(T, P)$  is then the usual process associated with  $T$  and the partition  $P$ . For each  $x$  we define its  $P_n$ -name to be  $P_n(x) : B_n \rightarrow P$  by  $P_n(x)(\vec{v}) = i$  if  $T_{\vec{v}}(x) \in p_i$ . To simplify our notation we will call an atom of  $\bigvee_{\vec{v} \in B_n} T_{\vec{v}} P$  of positive measure an  $n$ -name. The index  $\vec{0}$  in an  $n$ -name will be called the base point of the name.

We start with  $\pi : B_n \rightarrow B_n$ , a permutation of the indices in  $B_n$ , and define a size for this permutation. This idea is defined and extended in [1] and [4].

**Definition 2.1.** Let  $\pi : B_n \rightarrow B_n$  be a permutation of the indices of  $B_n$ . We say  $\pi$  is of size  $\epsilon$ , denoted by  $m(\pi) < \epsilon$ , if there exists a subset  $S$  of  $B_n$  satisfying

1.  $|S| > (1 - \epsilon)|B_n|$ , where  $|S|$  is the cardinality of the set  $S$ ,
2.  $\|\pi \vec{u} - \pi \vec{v} - (\vec{u} - \vec{v})\| < \epsilon \|\vec{u} - \vec{v}\|$  for every  $\vec{u}, \vec{v} \in S$ .

$S$  is said to be the  $\epsilon$ -set of  $\pi$ .

**Definition 2.2.** Given two  $P_n$ -names  $\eta$  and  $\xi$ , we define the  $\bar{f}_n$ -distance between them to be  $\bar{f}_n(\eta, \xi) = \inf \{\epsilon > 0 : \text{there exists a permutation } \pi \text{ of } B_n \text{ such that}$

- (i)  $m(\pi) < \epsilon$ ,
- (ii)  $\bar{d}(\eta \circ \pi, \xi) < \epsilon\}$ .

Here  $\bar{d}(\cdot, \cdot)$  denotes the Hamming metric which simply gives the proportion of locations of  $B_n$  on which the two names disagree.

Informally, we will think of  $\pi$  as rearranging the name  $\eta$  to make it  $\bar{d}$  close to the name  $\xi$ , and we will often refer to  $\pi$  as acting on a name instead of the (technically correct) set of indices.

We now define what appears to be a matching condition with a more rigid geometric requirement. We will show that in fact this matching requirement is closely related to  $\bar{f}$  matching.

**Definition 2.3.** For  $\epsilon > 0$ ,  $N \in \mathbb{N}$ , and  $0 < c < 1$ , two atoms  $\omega, \omega' \in \bigvee_{\vec{m} \in B_n} T_{\vec{m}} P$  are said to be  $(\epsilon, N, c)$  *matchable* if there exist a permutation  $\pi : B_n \rightarrow B_n$  and a set  $S$  of indices in  $B_n$  with the following properties:

1.  $|S| > (1 - \epsilon)|B_n|$ .
2.  $S$  is the disjoint union of  $N$ -blocks, call them  $\bigcup_{i=1}^k B_N + \vec{v}_i$ .
3.  $\pi$  moves all the indices in  $S$  by a vector  $\vec{v}$ , (possibly  $\vec{v} = \vec{0}$ ), small enough in magnitude that  $i + \vec{v} \in B_n$  for all  $i \in S$ .
4.  $\pi$  moves the  $N$  blocks in  $S$  by additional amounts which can vary for each block, but are always less in magnitude than  $\epsilon n$ .
5. In  $\omega \circ \pi$ , a subset of the  $N$ -blocks is matched perfectly with  $N$ -blocks in  $\omega'$ . Denote these matched blocks by  $G_i, i = 1, \dots, m$ . Then  $|\bigcup_{i=1}^m G_i| > c|B_n|$ . This set will be called the *matched set*  $G$  of  $\pi$ .
6. For  $\vec{u}, \vec{v} \in S$ , we have  $\|\pi(\vec{u}) - \pi(\vec{v}) - (\vec{u} - \vec{v})\| < \epsilon \|\vec{u} - \vec{v}\|$ .

Such a permutation  $\pi$  will be called an  $(\epsilon, N, c)$  *permutation*, or an  $(\epsilon, N, c)$  *match*. The set  $S$  will still be called the  $\epsilon$ -*set* of  $\pi$ .

Note that if  $\omega$  and  $\omega'$  are  $(\epsilon, N, c)$  matchable, then  $\bar{f}(\omega, \omega') < 1 - c$ . It is also true that  $\bar{f}$  closeness implies matchability in the above sense. To make this claim precise, we will use two facts about the geometry of a small permutation. First, if  $m(\pi) < \epsilon$ , then  $\frac{1}{\epsilon}$ -blocks in the  $\epsilon$ -set of  $\pi$  must be moved rigidly by  $\pi$ . The next fact requires a little more work, and the proof can be found in [4]:

**Lemma 2.4** (Geometric Lemma). *Given  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all permutations  $\pi : B_n \rightarrow B_n$  with  $m(\pi) < \delta$ , for all  $\vec{v}$  in the  $\delta$ -set of  $\pi$ ,*

$$\|\pi(\vec{v}) - \vec{v}\| < \epsilon n.$$

These two facts together yield that if two  $n$ -names are  $\bar{f}$ -close enough then they are matchable.

**Lemma 2.5.** *For all  $\epsilon > 0$  and  $N \in \mathbb{N}$ , there exist a  $\delta > 0$  and an integer  $n_1 > 0$  such that for all  $n \geq n_1$ , if  $\omega, \omega' \in \bigvee_{\vec{v} \in B_n} T_{\vec{v}} P$  and  $\bar{f}(\omega, \omega') < \delta$ , then  $\omega$  and  $\omega'$  are  $(\epsilon, N, 1 - \epsilon)$  matchable.*

*Proof.* Let  $\epsilon$  and  $N$  be given. Pick  $\delta_1 > 0$  satisfying Lemma 2.4 with this  $\epsilon$ . Let  $\delta = \min\{\delta_1, \frac{\epsilon}{4N^d}\}$  and  $n_1 > \frac{2dN}{\epsilon}$ . Pick  $n \geq n_1$  and suppose that  $\omega, \omega'$  are  $n$ -names with  $\bar{f}(\omega, \omega') < \delta$ . Let  $\pi : B_n \rightarrow B_n$  be a permutation such that  $m(\pi) < \delta$  and  $\bar{d}(\omega \circ \pi, \omega') < \delta$ . We will show that  $\pi$  satisfies the requirements of Definition 2.3.

If  $n^d \leq \frac{1}{\delta}$ , then  $\pi$  is the identity permutation and the result holds trivially. So suppose  $n$  is such that  $n^d > \frac{1}{\delta}$ . To find the set  $S$  required in Definition 2.3, we will use the portion of the  $\delta$ -set of  $\pi$  which can be easily divided into  $N$ -blocks. To find this, first let  $S_1$  be the  $\delta$ -set of  $\pi$  and note that the number of indices in  $B_n$  which are not in  $S_1$  or not matched by  $\pi$  is less than  $2\delta|B_n|$ . Then divide  $B_n$  into  $N$ -blocks, starting with the base point of the box, and let  $S$  be the union of those  $N$ -blocks which are completely contained in  $S_1$  and completely matched by  $\pi$ . Thus  $|S| \geq |B_n| - 2\delta|B_n|N^d - dNn^{d-1}$ , so  $|S| > (1 - \epsilon)|B_n|$ .

We also have that every  $\frac{1}{\delta}$ -block in  $S$ , so by our choice of  $\delta$  every  $N$ -block in  $S$ , is moved rigidly by  $\pi$  and matched perfectly to an  $N$ -block in  $\omega'$ . From the geometric lemma we are guaranteed that  $\forall \vec{v} \in S, \|\pi(\vec{v}) - \vec{v}\| < \epsilon n$ , so each  $N$ -block in  $S$  must be translated by some vector whose magnitude is less than  $\epsilon n$ . Thus conditions 1,

2, and 4 of Definition 2.3 are satisfied, and condition 3 is vacuously satisfied with  $\vec{v} = \vec{0}$ .

In this case the matched set of  $\pi$  is all of  $S$ , which is in the form  $\bigcup G_i$  by construction, and condition 5 is satisfied with  $c = 1 - \epsilon$ .

Finally, for condition 6, note that  $S \subset S_1$ , so for every pair  $\vec{u}, \vec{v} \in S_1$  we have  $\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| < \delta\|\vec{u} - \vec{v}\| < \epsilon\|\vec{u} - \vec{v}\|$ . All the conditions of  $(\epsilon, N, 1 - \epsilon)$  matchability are thus satisfied.  $\square$

### 3. THE LOOSELY BERNOULLI PROPERTY FOR ZERO ENTROPY $\mathbb{Z}^d$ ACTIONS

Intuitively, a zero entropy loosely Bernoulli process has one name up to  $\bar{f}$ . Formally,

**Definition 3.1.** A zero entropy, ergodic process  $(T, P)$  is *loosely Bernoulli* (LB) iff for any  $\epsilon > 0$  there exists an integer  $N_\epsilon$  such that for any  $n \geq N_\epsilon$  there is a set  $W \subset \bigvee_{\vec{v} \in B_n} T_{\vec{v}}P$  such that  $\mu(W) > 1 - \epsilon$  and, for  $\omega$  and  $\omega'$  in  $W$ ,

$$\bar{f}_n(\omega, \omega') < \epsilon.$$

**Definition 3.2.** We say  $(X, \mathcal{A}, \mu), T$  is LB if for every partition  $P$  of  $X$ ,  $(T, P)$  is LB.

Note that to show  $(X, \mathcal{A}, \mu), T$  is LB it suffices to show that the process  $(T, P)$  is LB for  $P$  a generating partition for  $T$ .

In the remainder of the section, we will define progressively weaker matching conditions which are in fact equivalent to LB. We will first state the new conditions and defer the proofs of the theorems to the end of the section.

**Definition 3.3** (The Matching Condition). A zero entropy process  $(T, P)$  is said to satisfy the *matching condition* if there is a  $c > 0$  such that for all  $\epsilon > 0$  and  $N \in \mathbb{N}$  there is an integer  $n_1 > 0$  such that for all integers  $n \geq n_1$ , there is a set of  $n$ -names  $W$  with  $\mu W > 1 - \epsilon$  and  $\forall \omega, \omega' \in W$ ,  $\omega$  and  $\omega'$  are  $(\epsilon, N, c)$  matchable.

If a process satisfies the matching condition then it is LB because, as the following result shows, once we can match a positive proportion of most names, we can keep matching.

**Theorem 3.4.** *If  $(T, P)$  is a zero entropy, ergodic,  $\mathbb{Z}^d$  process satisfying the matching condition, then  $(T, P)$  is LB.*

We can, in fact, prove that a process with an even weaker matching property is LB.

**Definition 3.5** (The Friendship Condition). A zero entropy process  $(T, P)$  is said to satisfy the *friendship condition* if there exist numbers  $c_1, c_2 > 0$  such that for all  $\epsilon > 0$  and  $N \in \mathbb{N}$  there is an integer  $n_1 > 0$  such that for all  $n \geq n_1$  there is a set of  $n$ -names  $W$  with  $\mu W > 1 - \epsilon$  and for all  $\omega \in W$  there is a set  $F(\omega)$  of  $n$ -names, with  $\mu(F(\omega)) \geq c_2$ , and for all  $\omega' \in F(\omega)$  we have that  $\omega$  and  $\omega'$  are  $(\epsilon, N, c_1)$  matchable.

We can show that a process satisfying the friendship condition is LB by arguing that if each name has a set of friends, then by sacrificing a little friendliness, we can find a large set of names which are all friendly with each other.

**Theorem 3.6.** *If  $(T, P)$  is a zero entropy, ergodic,  $\mathbb{Z}^d$  process satisfying the friendship condition, then  $(T, P)$  satisfies the matching condition.*

**Corollary 3.7.** *Let  $(T, P)$  be a zero entropy, ergodic,  $\mathbb{Z}^d$  process satisfying the friendship condition. Then  $(T, P)$  is LB.*

With these new results in place, showing that a process is LB will only require verifying that Definition 3.5 is satisfied.

**3.1. Proof of Theorem 3.4.** Fix  $\epsilon > 0$ . We will show that Definition 3.1 is satisfied with this  $\epsilon$ . By hypothesis, we know that there exists a number  $c > 0$  satisfying the matching condition (Definition 3.3). If  $c > 1 - \epsilon$ , then since  $(\epsilon, N, c)$  matchability of two  $n$ -names  $\omega, \omega'$  implies  $\bar{f}(\omega, \omega') < 1 - c < \epsilon$ , we are done.

Now suppose  $c < 1 - \epsilon$ . It suffices to show that we can find an  $n_2$  such that for every  $n \geq n_2$  there is a set  $W$  of  $n$ -names with measure larger than  $1 - \epsilon$ , and all the atoms in  $W$  are  $(\epsilon, N, c_1)$  matchable, with

$$c_1 = c + \frac{1}{2}c(1 - c).$$

For, if this is the case, we can keep matching until, at some stage  $k$ ,  $c_k > 1 - \epsilon$ .

Apply the matching condition with  $\frac{\epsilon^2}{100d}$  and arbitrary  $N \in \mathbb{N}$  to obtain an integer  $n_1$  and a set  $W_1$  of  $n_1$ -names all of which are  $(\frac{\epsilon^2}{100d}, N, c)$  matchable and for which

$$(1) \quad \frac{N}{n_1} < \frac{\epsilon}{4d}, \quad \text{and} \quad \mu W_1 > 1 - \frac{\epsilon^2}{100d}.$$

Take  $N_2 \in \mathbb{N}$  such that

$$(2) \quad \frac{n_1}{N_2} < \frac{\epsilon^2}{2^d \cdot 4d}.$$

Apply the matching condition to  $\frac{\epsilon^2}{100}$  and  $N_2$  and apply the ergodic theorem to  $W_1$  to obtain an integer  $n_2$  so that for all  $n \geq n_2$ , we can find a set  $W_2$  of  $n$ -names all of which are  $(\frac{\epsilon^2}{100}, N_2, c)$  matchable,

$$(3) \quad \mu W_2 > 1 - \frac{\epsilon^2}{50},$$

and

$$(4) \quad \text{for all } x \in W_2 \quad \frac{|\vec{v} \in B_n : T_{\vec{n}}x \in W_1|}{|B_n|} > 1 - \frac{\epsilon^2}{50}.$$

Take  $\omega, \omega' \in W_2$  and let  $\pi_1 : B_n \rightarrow B_n$  be an  $(\frac{\epsilon^2}{100}, N_2, c)$  permutation. Let  $S_1$  be the  $\frac{\epsilon^2}{100}$ -set of  $\pi_1$ . Consider now  $\omega \circ \pi_1$  and  $\omega'$ . We wish to match a subset of the indices left unmatched by  $\pi_1$ .

We first compute the proportion of unmatched indices of  $\omega \circ \pi_1$  which

- (i) lie in an  $N_2$  block from  $S_1$ ,
- (ii) are in  $W_1$ , and
- (iii) are such that the  $n_1$ -block based at that index is completely contained inside the unmatched  $N_2$  block to which the index belongs.

By condition 1 of Definition 2.3, equation (4), and the fact that there are at least  $\epsilon n^d$  unmatched indices, the above conditions eliminate a set of indices of cardinality less than

$$\frac{\epsilon^2}{100}n^d + \frac{\epsilon^2}{50}n^d + (dn_1N_2^{d-1})(\# \text{ of unmatched } N_2 \text{ blocks}) < \frac{\epsilon}{10}n^d.$$

Similarly, we can assume that all but an  $\frac{\epsilon}{10}$  proportion of the unmatched indices in  $\omega'$  will also satisfy conditions (i), (ii) and (iii).

Consider the set  $\{r_{\vec{v}} : \vec{v} \in B_{n_1}\}$  of base points of all  $n_1$ -grids of  $B_n$ . For each  $\vec{v} \in B_{n_1}$ , let  $\bar{r}_{\vec{v}}$  be that portion of  $r_{\vec{v}}$  which is contained in the unmatched indices. Since  $\bigcup_{\vec{v} \in B_{n_1}} \bar{r}_{\vec{v}}$  is the entire collection of unmatched indices, we can find an  $n_1$ -grid of the unmatched part of  $B_n$  such that all but  $\frac{\epsilon}{5}$  of the base points of the grid boxes for both  $\omega \circ \pi$  and  $\omega'$  satisfy conditions (i), (ii) and (iii). Fix this grid superimposed on both  $\omega \circ \pi_1$  and  $\omega'$ , and call a grid box with such a base point a good grid box. For a pair of good grid boxes in the same location in the two names, apply the  $(\frac{\epsilon^2}{100d}, N, c)$  match guaranteed by the definition of  $W_1$ . Let  $\pi$  be the permutation obtained by first applying  $\pi_1$ , followed by the individual permutations applied to good  $n_1$ -grid boxes.

We now show that  $\pi$  is an  $(\epsilon, N, c_1)$  permutation. Note that the  $j^{th}$  good grid box comes with an  $\frac{\epsilon^2}{100d}$ -set  $S(j)$  and a matched set  $G^j$  consisting of matched  $N$ -blocks.

The  $\epsilon$ -set  $S$  of  $\pi$  will consist of all the indices in  $S_1$  except for

1. those who lie in an  $n_1$ -grid box which is not entirely contained in an  $N_2$  box, and
2. those which are in a good grid box but
  - do not belong to  $S(j)$ , or
  - are in an  $N$ -block which is not contained in the  $\frac{\epsilon}{4d}$ -interior of its grid box.

This removes a set of indices with cardinality less than

$$dn_1N_2^{d-1}(\# \text{ unmatched } N_2 \text{ blocks}) + (\frac{\epsilon^2}{100d}n_1^d + \frac{\epsilon}{4d}n_1^d + 2dNn_1^{d-1})(\# \text{ good grid boxes}),$$

which by equations (1) and (2) is less than  $\epsilon n^d$ . Then  $|S| > (1 - \epsilon)|B_n|$ , so condition 1 of Definition 2.3 is satisfied.

Since, without loss of generality, we can assume  $N_2$  and  $n_1$  are multiples of  $N$ , condition 2 is satisfied.

Condition 3 is satisfied because the indices in  $S$  are also in  $S_1$ , the  $\frac{\epsilon^2}{100}$ -set of  $\pi_1$ , and we use for  $\vec{v}$  the vector from  $\pi_1$ .

Note that condition 4 is automatically satisfied for the  $N_2$  blocks moved by  $\pi_1$ . The bad  $n_1$ -grid boxes have no additional translation applied to them. Consider now the  $N$ -blocks in the good  $n_1$ -grid boxes. In addition to the translation by  $\pi_1$ , which is in magnitude less than  $\frac{\epsilon^2}{100}n$ , these may have been moved again by  $\pi$ . In particular, considering conditions 3 and 4 of Definition 2.3, their individual match will have possibly moved each by additional vectors of magnitude  $< \frac{2\epsilon^2}{50d}n_1$ . This is a total displacement of size less than  $\epsilon n$ .

For condition 5 we define the matched set  $G$  of  $\pi$  to be the following collection of  $N$  boxes:

- (i) the matched set of  $\pi_1$ , i.e. the  $N_2$  blocks matched by  $\pi_1$ , and
- (ii) the  $N$ -blocks in  $G^j \cap S$ .

The collection of indices satisfying (i) is by hypothesis larger than  $cn^d$ . For (ii), note that the  $G^j$  were such that each  $|G^j| > cn_1^d$ . In restricting to  $S$ , we throw away less than  $2d\frac{\epsilon}{4d}n_1^d + 2dNn_1^{d-1}$  indices in each good  $n_1$ -grid box in order to consider only those  $N$  blocks entirely contained in the  $\frac{\epsilon}{4d}$ -interior of the  $n_1$ -grid box. This is less than  $\epsilon$  of each good  $n_1$ -grid box. So we have included at least  $(c - \epsilon)$  proportion

of each good grid box in the matched set of  $\pi$ . Thus of the part left unmatched by  $\pi_1$  we have matched a proportion no less than

$$(c - \epsilon)(\text{proportion of unmatched indices of } \omega \circ \pi_1 \text{ in good grid boxes}).$$

Now we count the number of unmatched indices of  $\omega \circ \pi_1$  in good grid boxes. An index not in a good grid box either is not in the grid we are considering, or is in a bad grid box. This is a set of indices of cardinality less than

$$2dn_1n^{d-1} + 2d(2n_1 + N_2)^{d-1}n_1(\text{number of matched } N_2 \text{ boxes}) \\ + (\text{number of bad base points})n_1^d.$$

Using equation (2), the fact that there are at least  $\epsilon n^d$  unmatched indices, and our previous calculations about bad base points in the grid, we have that this is a proportion less than  $\frac{\epsilon}{2} + \frac{\epsilon}{5}$  of the unmatched indices. Thus the proportion of unmatched indices of  $\omega \circ \pi_1$  in good grid boxes is at least  $1 - \frac{7\epsilon}{10}$ .

Putting this all together, we have matched an additional proportion of at least  $(c - \epsilon)(1 - \frac{7\epsilon}{10})$ . For small enough  $\epsilon$ , this is larger than  $\frac{1}{2}c$ . So after applying the permutation  $\pi$ , we have matched at least  $c + \frac{1}{2}c(1 - c) = c_1$  proportion of the indices, satisfying condition 5 of  $(\epsilon, N, c_1)$  matchability.

Finally, we show that condition 6 is satisfied. Note that every  $\vec{u}, \vec{v}$  in  $S$  (the  $\epsilon$ -set of  $\pi$ ) is in  $S_1$ , the  $\frac{\epsilon^2}{100}$ -set of  $\pi_1$ . In particular, if  $\vec{u}$  and  $\vec{v}$  are both only moved by  $\pi_1$ , then condition 6 holds automatically.

The only difficulty then arises if one or both of  $\vec{u}$  and  $\vec{v}$  in  $S$  have been moved by an individual  $n_1$ -permutation. Suppose  $\vec{u}$  is such an index. There are several cases to consider:

- (i)  $\vec{v}$  lies in an  $N_2$  box matched by  $\pi_1$ , or
- (ii)  $\vec{v}$  lies in a different  $n_1$ -box from  $\vec{u}$ , or
- (iii)  $\vec{v}$  lies in the same  $n_1$ -box as  $\vec{u}$ .

In cases (i) and (ii) we will use the fact that for such a  $\vec{u}, \vec{v}$ ,

$$(5) \quad \|\vec{u} - \vec{v}\| > \frac{\epsilon}{4d}n_1.$$

In case (i) note that  $\pi(\vec{v}) = \pi_1(\vec{v})$ , so

$$\|\vec{u} - \vec{v} - (\pi\vec{u} - \pi\vec{v})\| \leq \|\vec{u} - \vec{v} - (\pi_1\vec{u} - \pi_1\vec{v})\| + \|\pi_1\vec{u} - \pi_1\vec{v} - (\pi_2\pi_1\vec{u} - \pi_1\vec{v})\|,$$

where  $\pi_2$  is the  $(\frac{\epsilon^2}{100d}, N, c)$ -permutation which affects  $\vec{u}$ . Since  $\vec{u}$  and  $\vec{v}$  are in  $S_1$ , the first term is less than  $\frac{\epsilon^2}{100}\|\vec{u} - \vec{v}\|$ . Note that condition 1 of Definition 2.3 implies that the magnitude of the vector in condition 3 is less than  $\frac{\epsilon^2}{100d}n_1$ . So by conditions 3 and 4 combined,  $\pi_2$  will have moved  $\vec{u}$  by less than  $\frac{\epsilon^2}{50d}n_1$ . Now by equation (5) we have

$$\|\vec{u} - \vec{v} - (\pi\vec{u} - \pi\vec{v})\| < \frac{\epsilon^2}{100}\|\vec{u} - \vec{v}\| + \frac{4\epsilon}{50}\|\vec{u} - \vec{v}\| < \epsilon\|\vec{u} - \vec{v}\|.$$

In case (ii),

$$\|\vec{u} - \vec{v} - (\pi\vec{u} - \pi\vec{v})\| \leq \frac{4\epsilon^2}{100d}n_1,$$

because of conditions 3 and 4 of  $\frac{\epsilon^2}{100d}$ -matchability. Again by equation (5) we have

$$\|\vec{u} - \vec{v} - (\pi\vec{u} - \pi\vec{v})\| \leq \epsilon\|\vec{u} - \vec{v}\|.$$



In case (iii), by construction  $\vec{u}$  and  $\vec{v}$  will lie in the  $\frac{\epsilon^2}{100d}$ -set of the same  $n_1$ -permutation. Hence condition 6 holds, and we are done.

**3.2. Proof of Theorem 3.6.** Let  $c_1$  and  $c_2$  be the constants from Definition 3.5, and pick  $c < c_1 \cdot c_2$ . We will show that the matching condition is satisfied with this  $c$ . The proof of the result for a given  $\epsilon, N$  is divided into three parts. First we define the set  $W$ , then for a pair  $\omega, \omega' \in W$ , we define a permutation  $\pi_\omega$ , and finally we show that  $\pi_\omega$  is an  $(\epsilon, N, c)$ -permutation.

Start by finding nonzero  $\epsilon_1, \epsilon_2$  such that  $c = (c_1 - \epsilon_1)(c_2 - \epsilon_2)$ , and then set  $\gamma = \min\{\frac{1}{d}\epsilon, \epsilon_1, \frac{1}{d}\epsilon_2\}$ . Apply the friendship condition with  $\frac{\gamma^2}{100}$  and an arbitrary  $N$  to find an integer  $n_0$  large enough that

$$(6) \quad \frac{N}{n_0} < \frac{\gamma}{16}$$

and there exists a set  $W_{n_0}$  of  $n_0$ -names with  $\mu(W_{n_0}) > 1 - \frac{\gamma^2}{100}$  such that every  $\omega \in W_{n_0}$  has a set of friends,  $F(\omega)$ .

Now apply the pointwise ergodic theorem to obtain an integer  $n_1$  large enough that there is a set  $U_{n_1}$  of  $n_1$ -names with

$$\mu(U_{n_1}) > 1 - \frac{\gamma}{100}$$

and for all  $x \in U_{n_1}$  and all  $n_0$ -names  $\omega$ ,

$$\frac{|\vec{v} \in B_{n_1} : T_{\vec{v}}x \in \omega|}{|B_{n_1}|} \in \left( \mu(\omega) - \frac{\gamma}{100}, \mu(\omega) + \frac{\gamma}{100} \right).$$

Applying the ergodic theorem again, we obtain  $n_2$  such that for  $n \geq n_2$  there is a set of  $n$ -names  $W$  with  $\mu(W) \geq 1 - \frac{\gamma}{50}$ , such that for  $x \in W$  we have

$$\frac{|\vec{v} \in B_n : T_{\vec{v}}x \in U_{n_1}|}{|B_n|} > 1 - \frac{\gamma}{50} \quad \text{and} \quad \frac{|\vec{v} \in B_n : T_{\vec{v}}x \in W_{n_0}|}{|B_n|} > 1 - \frac{\gamma}{50}.$$

Now take  $n > n_2$  such that

$$(7) \quad \frac{d(n_1 + 3n_0)}{n} < \frac{\gamma}{100},$$

and consider the set  $W$  as described above. We will show that  $W$  satisfies the statement of the theorem.

Take  $\omega, \omega' \in W$ . We will now construct an  $(\epsilon, N, c)$ -permutation for this pair.

Consider those indices in  $\omega$  which lie in  $B_{n-(n_1+2n_0)}$ . Note that by equation (7) we have

$$(8) \quad \frac{|B_{n-(n_1+2n_0)}|}{|B_n|} > 1 - \frac{\gamma}{100}.$$

Hence the proportion of  $B_{n-(n_1+2n_0)}$  in  $\omega$  which is not in  $W_{n_0}$  must be less than  $\frac{\gamma}{25}$ .

Now consider all  $R_{\vec{v}}$  with  $\vec{v} \in B_{n_0}$ , the  $n_0$ -grids of  $B_{n-(n_1+2n_0)}$ , and their base points  $r_{\vec{v}}$ . Since  $B_{n-(n_1+2n_0)} = \bigcup_{\vec{v} \in B_{n_0}} R_{\vec{v}}$ , there must be a grid with all but  $\frac{\gamma}{25}$  of its base points in  $W_{n_0}$ .

Fix this grid  $R_{\vec{v}}$ , and draw the identical grid on  $\omega'$ . Note that

$$(9) \quad |R_{\vec{v}}| > |B_n| - d(3n_0 + n_1)n^{d-1}.$$

Using this with (7) and the properties of  $W$ , we have that in  $\omega'$  the proportion of indices in this grid which are not locations of  $U_{n_1}$  is less than  $\frac{\gamma}{25}$ . Thus there is a vector  $\vec{r} \in B_{n_0}$  such that for at least

$$(10) \quad 1 - \left(\frac{\gamma}{25} + \frac{\gamma}{25}\right) = 1 - \frac{2\gamma}{25}$$

of the grid boxes:

1. in  $\omega'$  this location is an occurrence of  $U_{n_1}$ , and
2. in  $\omega$  this is a location in an  $n_0$ -grid box whose base point is an occurrence of  $W_{n_0}$ .

Call such an  $n_0$ -grid box in  $\omega$  a “good” box, and call the  $n_1$ -name in  $\omega'$  in location  $\vec{r}$  its associated  $n_1$ -name. For a good grid box its associated  $n_1$ -name is at least  $c_1 - \frac{\gamma}{100}$  full of its friends. Thus one location in  $B_{n_1}$ , say  $\vec{m}$ , is such that for  $c_1 - \frac{\gamma}{100}$  of the good boxes, the location  $\vec{m} + \vec{r}$  in its associated  $n_1$ -name is the base point of a friend. Let  $G$  be this subset of good boxes in  $\omega$ . Then by equations (7),(9) and (10) we have

$$(11) \quad |G| > (c_1 - \gamma)|B_n|.$$

Finally we can define  $\pi_\omega$ . First define  $\pi_1 : B_n \rightarrow B_n$  by  $\pi_1(\vec{v}) = \vec{v} + \vec{r} + \vec{m}$  for  $\vec{v} \in B_{n-(n_0+n_1)}$ . The indices in the edges of  $B_n$  which are dislocated by the translation will be moved by  $\pi_1$  to arbitrary indices vacated by the translation. Notice that after applying  $\pi_1$  to  $\omega$ , the  $n_0$ -names in  $G$  are lined up with a friend in  $\omega'$ . Now define  $\pi_\omega : B_n \rightarrow B_n$  by first applying  $\pi_1$  and then, on the  $n_0$ -names in  $G$ , applying the permutation given by the  $(\frac{\gamma^2}{100}, N, c_2)$  matchability of the two friends.

We now show that  $\pi_\omega$  is the permutation that satisfies  $(\epsilon, N, c)$  matchability. Recall that  $G$  consists of  $n_0$ -grid boxes in  $\omega$  which line up with friends after applying  $\pi_1$ . Each such  $n_0$ -name has a  $\frac{\gamma^2}{100}$ -set  $S_i$  which is the union of  $N$ -boxes, and  $|S_i| > (1 - \frac{\gamma^2}{100})|B_{n_0}|$ . Let  $\tilde{S}_i$  be the union of those  $N$ -boxes which lie entirely in the  $\frac{\gamma}{16}$  interior of  $B_{n_0}$ . We have thus eliminated at most  $2d(\frac{\gamma}{16}n_0 + N)n_0^{d-1}$  indices from  $S_i$ , which by (6) is less than  $\frac{d\gamma}{4}n_0^d$ . Thus  $|\tilde{S}_i| > (1 - \frac{\gamma^2}{100} - \frac{d\gamma}{4})|B_{n_0}|$ . Now set  $S$  to be the union of these  $\tilde{S}_i$  plus the  $n_0$ -boxes of the grid not in  $G$ . Then by the above calculation, equation (8), and our choice of  $\gamma$ , we have  $|S| > (1 - \epsilon)|B_n|$ . This gives us condition 1 of Definition 2.3.

Now take  $\vec{u}, \vec{v} \in S$ . If  $\vec{u}, \vec{v}$  lie in the same  $n_0$ -box, by construction

$$\|\pi(\vec{u}) - \pi(\vec{v}) - (\vec{u} - \vec{v})\| < \frac{\gamma^2}{100}\|\vec{u} - \vec{v}\| < \epsilon\|\vec{u} - \vec{v}\|.$$

Otherwise, suppose  $\vec{u}, \vec{v}$  are from distinct  $n_0$ -boxes. If both are  $n_0$ -boxes not in  $G$ , then

$$\|\pi(\vec{u}) - \pi(\vec{v}) - (\vec{u} - \vec{v})\| = 0.$$

If at least one  $n_0$ -box is in  $G$ , we have from  $(\frac{\gamma^2}{100}, N, c_2)$  matchability that  $\|\pi(\vec{u}) - \pi(\vec{v}) - (\vec{u} - \vec{v})\| < 2\frac{\gamma^2}{100}n_0$ . Because we are only considering the  $\frac{\gamma}{16}$  interior of the  $n_0$ -boxes in  $G$ , we also have  $\|\vec{u} - \vec{v}\| \geq \frac{\gamma}{16}n_0$ . So

$$\|\pi(\vec{u}) - \pi(\vec{v}) - (\vec{u} - \vec{v})\| < \epsilon\|\vec{u} - \vec{v}\|,$$

and condition 6 is satisfied.

Without loss of generality we can assume  $n_0$  is a multiple of  $N$ , and thus condition 2 is also satisfied.

Next notice that all the indices in  $S$  were first shifted by the vector  $\vec{m} + \vec{r}$  satisfying  $\|\vec{m} + \vec{r}\| < n_0 + n_1$ . Since  $S$  is contained in the subbox of  $B_n$  which is a distance  $n_0 + n_1$  from the edge of  $B_n$ , condition 3 of Definition 2.3 is satisfied.

The permutation  $\pi$  further moves the  $N$ -boxes found in  $G$  by amounts given by the  $(\frac{\gamma^2}{100}, N, c_2)$  matchability of their respective  $n_0$ -boxes. Their total translation is thus less than  $n_0 + \frac{\gamma^2}{100}n_0$ , which by (7) and our choice of  $\gamma$  is less than  $\epsilon n$ , as required for condition 4.

Recall that  $S_i$  was the  $\frac{\gamma^2}{100}$ -set of a good box and  $\tilde{S}_i \subset S_i$  was the subset of  $S_i$  of  $N$ -blocks which were entirely in the  $\frac{\gamma}{16}$ -interior of  $B_{n_0}$ . Let  $\bigcup G_i$  be the union of those  $N$ -blocks in all the  $\tilde{S}_i$ 's which were matched perfectly by the  $(\frac{\gamma^2}{100}, N, c_2)$  matchability. Then  $|\bigcup G_i| > (c_2 - \frac{d\gamma}{4})|G|$ , and hence, by (11) and our choice of  $\gamma$ ,  $|\bigcup G_i| > (c_2 - \frac{d\gamma}{4})(c_1 - \gamma)|B_n| > c|B_n|$ .

Since conditions 1 through 6 of Definition 2.3 are satisfied, we have that  $\omega, \omega' \in \hat{W}$  are  $(\epsilon, N, c)$  matchable, as wanted.

#### 4. $k$ POINT EXTENSIONS

Let  $k \geq 2$  be an integer. To define a  $k$ -point extension of the ergodic, zero entropy  $\mathbb{Z}^d$  action  $(X, \mu, T)$  we let  $\{c_1, \dots, c_k\}$  denote the discrete space with  $k$  points. We think of each  $c_i$  as representing a different color. Let  $\overline{X} = X \times \{c_1, \dots, c_k\}$ ,  $\overline{\mathcal{A}} = \mathcal{A} \times 2^{\{c_1, \dots, c_k\}}$ , and  $\overline{\mu}(A \times \{c\}) = \frac{1}{k}\mu(A)$ , where  $A \in \mathcal{A}$  and  $c \in \{c_1, \dots, c_k\}$ .

Let  $S_k$  denote the symmetric group on  $k$  symbols. Let  $h : X \times \mathbb{Z}^d \rightarrow S_k$  be a measurable  $T$  cocycle. So for every  $\vec{m}, \vec{n} \in \mathbb{Z}^d$  we have

$$h(x, \vec{m} + \vec{n}) = h(x, \vec{m}) \circ h(T_{\vec{m}}(x), \vec{n}).$$

We can then define a  $\mathbb{Z}^d$ -action  $\{T_{\vec{v}}^h\}$  in the following way:

$$T_{\vec{v}}^h(x, c) = (T_{\vec{v}}(x), h_{\vec{v}}(x)c).$$

For any such extension, if  $T$  has zero entropy then  $T^h$  also has zero entropy. If  $P_1$  is a generating partition for  $T$ , then  $P = P_1 \vee \{\{c_1\}, \dots, \{c_k\}\}$  is a generating partition for  $T^h$ . We call it the extension of  $P_1$ . For the remainder of this section we fix  $P_1$ , a generating partition for  $T$ , and set  $P$  as above.

For  $1 \leq i \leq d$  we denote the measurable functions  $h : X \times \{\vec{e}_i\}$  by  $h_i$ . We say the extension  $(\overline{X}, \overline{\mathcal{A}}, \overline{\mu}), T^h$  is trivial if  $h_i$  is constant for every  $i$ .

In this section we will first show that if  $(T^h, P)$  is an ergodic trivial extension of a zero entropy LB system, then  $(T^h, P)$  is LB. We will then prove the result for a non-trivial  $k$ -point extension by reducing the argument to the trivial case.

**Theorem 4.1.** *If  $T$  is an ergodic, zero entropy and LB  $\mathbb{Z}^d$  action, and  $T^h$  is an ergodic trivial  $k$ -point extension of  $T$ , then  $T^h$  is LB.*

*Proof.* Let  $P$  be the extension of a generating partition of  $T$ . Since  $P$  is a generating partition for  $T^h$ , by Corollary 3.7 it suffices to show that  $(T^h, P)$  satisfies the friendship condition. We will show that  $(T^h, P)$  satisfies Definition 3.5 with  $c_1 = c_2 = \frac{1}{2k}$ .

Let  $0 < \epsilon < \frac{1}{2k}$  and  $N \in \mathbb{N}$  be fixed. Find  $\delta < \epsilon$  and  $n_1 \in \mathbb{N}$  satisfying Lemma 2.5. Since  $(X, T)$  is LB, we can assume  $n_1$  is such that for every  $n > n_1$  and  $\delta$ -a.e.  $n$ -names  $\omega$  and  $\omega'$ ,  $\overline{f}_n(\omega, \omega') < \delta$ . Let this set of atoms be the set  $W_n$ , and put  $\overline{W}_n = \{\overline{\omega} : \overline{\omega} \text{ is an extension of } \omega \in W_n\}$ . For  $\overline{\omega} \in \overline{W}_n$ , we will construct a set

$F(\bar{\omega})$ . We will do this by showing that every other atom in  $W_n$  has an extension which belongs to  $F(\bar{\omega})$ . This will be done by first finding the proposed extension, showing that the collection of such things has sufficient measure, and then showing that indeed such extensions can be matched to  $\bar{\omega}$ .

So fix  $\bar{\omega}$ , an extension of  $\omega$ , and let  $\omega'$  be another point in  $W_n$ . By Lemma 2.5 we have that  $\omega, \omega'$  are  $(\epsilon, N, 1 - \epsilon)$  matchable. Let  $\pi$  be the permutation satisfying the matchability conditions.

Consider  $G$ , the matched set of  $\pi$ . Say  $G = \bigcup G_i$ , where each  $G_i$  is a matched  $N$ -block. Denote their counterparts in  $\omega'$  by  $\{G'_i\}$  and the corresponding colored boxes in  $\bar{\omega}$  by  $\{\bar{G}_i\}$ . We know that  $\omega'$  has exactly  $k$  extensions and the color at every index is different in different extensions. Call the extensions  $\bar{\omega}'_j$  for  $j = 1, \dots, k$ . We denote the corresponding colorings of  $G'_i$  by  $\bar{G}'_{i,j}$ .

Look at the color at the lower left hand corner of box  $\bar{G}_i$  in  $\bar{\omega}$ . In one of the  $\bar{G}'_{i,j}$ , the color at the lower left hand corner must be the same as in  $\bar{G}_i$ ; hence the entire  $N$ -box must have the same coloring as in  $\bar{G}_i$ . This is true for every  $i$ , so there is a  $j \in \{1, \dots, k\}$  such that for at least  $\frac{1}{k}$  of the boxes  $\bar{G}_i$ ,  $\pi$  matches  $\bar{G}_i$  perfectly with  $\bar{G}'_{i,j}$  in  $\bar{\omega}'_j$ . Call such an extension of  $\omega'$  a good extension, and set  $F(\bar{\omega}) = \{\bar{\omega}' \in \bar{W}_n : \bar{\omega}' \text{ is a good extension of } \omega' \in W_n\}$ .

By the argument above it is clear that

$$\bar{\mu}(F(\bar{\omega})) \geq \frac{1}{k} \bar{\mu}(\bar{W}_n) > \frac{1}{k}(1 - \delta) > \frac{1}{k} - \frac{\epsilon}{k} > \frac{1}{2k},$$

as desired.

We now want to show that, for  $\bar{\omega}' \in F(\bar{\omega})$ ,  $\bar{\omega}$  and  $\bar{\omega}'$  are  $(\epsilon, N, \frac{1}{2k})$  matchable.

Let  $S$  and  $\pi$  be as defined by the matchability on the base space, so all conditions of Definition 2.3 except condition 5 are automatically satisfied. To see that condition 5 holds, note that the indices in  $\bar{G}$ , the matched set of  $\pi$  applied to the extension, is a subset of the indices in  $G$ , the original matched set. Our earlier argument shows that

$$|\bar{G}| \geq \frac{1}{k}|G| > \frac{1}{k}(1 - \epsilon)|B_n|,$$

so condition 5 is satisfied with  $c = \frac{1}{2k}$ .  $\square$

**Theorem 4.2.** *If  $T$  is an ergodic, zero entropy and LB  $\mathbb{Z}^d$  action, and  $T^h$  is an ergodic  $k$ -point extension of  $T$ , then  $T^h$  is LB.*

*Proof.* Let  $P_1$  be a generating partition for  $T$ , and for  $1 \leq i \leq d$  let  $E_i = \{h_{\bar{e}_i}^{-1}(\sigma)\}_{\sigma \in S_k}$ . The partition  $P_2 = P_1 \vee E_1 \vee \dots \vee E_d$  is also a measurable, generating partition for  $T$ ; hence its extension  $P$  is generating for  $T^h$ . Since  $T$  is LB, then  $(T, P_2)$  is also LB.

Note that with the partition  $P$ , every  $P_2$ -name  $\omega$  has exactly  $k$  extensions, the color at every index is different for different extensions, and knowing the color of one index determines the color of a whole box. In fact, with this partition we can treat the extension as though it were a trivial extension. The argument now follows as in the proof of Theorem 4.1.  $\square$

## 5. ISOMETRIC EXTENSIONS

In this section we let  $(C, \rho)$  be a compact, homogeneous metric space and  $G$  be the group of all isometries of  $C$ . Note that  $G$  is then a compact group [3]. Let  $m$  be

the  $G$ -invariant measure on  $C$ , and  $(X, \mu, T)$  a free, measure preserving, ergodic, zero entropy  $\mathbb{Z}^d$  action. Suppose  $h : X \times \mathbb{Z}^d \rightarrow G$  is a measurable  $T$  cocycle. Namely, for all  $\vec{n}, \vec{m} \in \mathbb{Z}^d$  we have that

$$(12) \quad h(x, \vec{n} + \vec{m}) = h(x, \vec{n}) \circ h(T_{\vec{n}}x, \vec{m}).$$

If for  $\vec{n} \in \mathbb{Z}^d$  we define  $T^h : X \times C \rightarrow X \times C$  by

$$T_{\vec{n}}^h(x, c) = (T_{\vec{n}}x, h(x, \vec{n})(c)),$$

then by equation (12),  $T^h$  will be a measure preserving  $\mathbb{Z}^d$  action on  $(X \times C, \mu \times m)$ .  $T^h$  will have the same entropy as  $T$  [3]. We will refer to  $T^h$  as a compact group extension of  $T$ .

**Theorem 5.1.** *Let  $(X, \mu, T)$  be a free, ergodic and measure preserving, zero entropy, and LB  $\mathbb{Z}^d$  action. Let  $(C, \rho)$  be a compact, homogeneous metric space and  $G$  be the group of all isometries of  $C$ . Let  $h : X \times \mathbb{Z}^d \rightarrow G$  be a  $T$  cocycle, and suppose  $T^h$  is an ergodic isometric extension of  $T$ . Then  $T^h$  is LB.*

*Proof.* Let  $T$  and  $h$  be as above. We abbreviate  $h_i(x) = h(x, \vec{e}_i)$  for each dimension  $1 \leq i \leq d$ . Denote  $\mu \times m$  by  $\hat{\mu}$ . Take a partition  $P$  of  $X$  which is a generating partition for  $T$ , and a partition  $Q$  on  $C$  such that the topological boundary of  $Q$  has measure zero, and such that  $P \times Q$  is a generating partition for  $T^h$ . We now show that  $(T^h, P \times Q)$  is LB. Since  $T$  is zero entropy so is  $T^h$ . Thus we need to prove that  $(T^h, P \times Q)$  satisfies Definition 3.1.

We will do two simplifications: we will approximate  $Q$  by a simpler partition  $R$ , and we will approximate each  $h_i$  by a step function. The outline of the proof is as follows. We will initially use the partition  $R$  and a weakened sense of matching of  $P^L \times R$  names to find modified friends for a large set of atoms, where  $P^L$  is a refinement of  $P$  to be determined later and where we will later make the term “modified friends” precise. We will then use our previous machinery to proceed to a large set of atoms all of which are modified friends, and then to conclude a modified form of LB. Finally, we will remove this modification and prove that  $T^h$  is LB.

Fix  $\epsilon > 0$  and  $N \in \mathbb{N}$ . Suppose  $R$  is a partition of the space  $C$  into  $r$  elements, and label the sets of  $R$  initially in arbitrary order by  $\{R_1, R_2, \dots, R_r\}$ . Let

$$\begin{aligned} d_1 &= \min_{i \in \{1, \dots, r\}} \{\text{diam}(R_i)\}, & d_2 &= \max_{i \in \{1, \dots, r\}} \{\text{diam}(R_i)\}, \\ \beta_1 &= \min_{i \in \{1, \dots, r\}} \{m(R_i)\}, & \beta_2 &= \max_{i \in \{1, \dots, r\}} \{m(R_i)\}. \end{aligned}$$

For the alphabet of the partition  $R$ , construct a labeling scheme so that by reading the label assigned to set  $R_i$  we can identify all sets  $R_j$  such that

$$(13) \quad \rho(R_i, R_j) \leq 2d_2.$$

We will say that two sets  $R_i, R_j$  which satisfy (13) are adjacent.

Let

$$(14) \quad \epsilon_1 < \min \left\{ \frac{\epsilon}{12Nd}, \frac{d_2}{Nd} \right\}$$

and choose  $L > 0$  so large that there are functions  $\tilde{h}_i : X \rightarrow G$  such that

1. if  $\omega \in \bigvee_{\vec{v} \in B_L} T_{\vec{v}}P$ , then  $\tilde{h}_i(x) = g_i$  for every  $x \in \omega$ ,  $1 \leq i \leq d$ , and
2. for  $\epsilon_1$ -almost every  $x \in X$ ,  $\rho(h_i(x), \tilde{h}_i(x)) < \epsilon_1$ ,  $1 \leq i \leq d$ .

We will now use the partition  $P^L = \bigvee_{\vec{v} \in B_L} T_{\vec{v}}P$  instead of  $P$ . Since  $P^L$  is a refinement of  $P$ ,  $P^L \times Q$  is also a generating partition for  $T^h$  and it suffices to show LB for this partition.

Let  $B = \{x : \rho(h_i(x), \tilde{h}_i(x)) > \epsilon_1 \text{ for some } 1 \leq i \leq d\}$ . Then  $\mu(B) < \epsilon_1$ .

Applying loosely Bernoulli and the ergodic theorem to the base, we obtain  $n > 0$  and a set of atoms  $W_n$  such that

$$(15) \quad \mu(W_n) > 1 - \frac{\epsilon}{3},$$

$$(16) \quad \omega, \omega' \in W_n \text{ are } \left(\frac{\epsilon}{3}, N, 1 - \frac{\epsilon}{3}\right) \text{ matchable,}$$

and the set  $H = \{x : |\vec{v} \in B_n : T_{\vec{v}}x \in B|/|B_n| < 2\epsilon_1\}$  has measure

$$(17) \quad \mu H > 1 - \frac{\epsilon^2}{18} \frac{\beta_1}{4\beta_2}.$$

Next, consider the atoms of  $W_n$ . We will call an atom  $\omega$  “good” if

$$(18) \quad \mu(\omega \cap H^c) < \frac{1}{4} \frac{\beta_1}{\beta_2} \frac{\epsilon}{6} \mu(\omega).$$

Consider only those atoms in  $W_n$  which are good, and still call this new set  $W_n$ . Then

$$(19) \quad \mu(W_n) > 1 - \frac{\epsilon}{3} - \frac{\epsilon}{3}.$$

Fix an atom  $\omega$  from  $W_n$  and consider all the  $P^L \times R$ ,  $n$ -names which have  $\omega$  as their  $P^L$  label. We call these the extension atoms of  $\omega$ . We want to consider only those extension atoms which are substantially covered by the set  $H \times C$ . We define an extension-atom  $\hat{\omega}$  to be “bad” if  $\hat{\mu}(\hat{\omega} \cap (H^c \times C)) > \frac{1}{2} \hat{\mu}(\hat{\omega})$ . Since every  $\omega \in W_n$  is a good atom in the base space, we have that for a fixed good base atom  $\omega$ , the measure of all the bad extension atoms of  $\omega$  is less than  $\frac{\epsilon}{3} \mu(\omega)$ .

Let  $\hat{W}_n = \bigcup_{\omega \in W_n} \{\hat{\omega} : \hat{\omega} \text{ is a good extension of } \omega\}$ . Then

$$\mu(\hat{W}_n) > (1 - \frac{\epsilon}{3})(1 - \frac{\epsilon}{3} - \frac{\epsilon}{3}) > (1 - \epsilon).$$

For the rest of our work with  $P^L \times R$  names we will modify our notion of matching. Let  $\hat{\omega}$  and  $\hat{\omega}'$  be two  $P^L \times R$  names. We will say that the  $R$ -labels  $i$  and  $j$  at any index  $\vec{v}$  in  $\hat{\omega}$  and  $\hat{\omega}'$  agree if equation (13) is satisfied for this  $R_i$  and  $R_j$ .

We now fix  $\hat{\omega} \in \hat{W}_n$  and let  $\omega$  be its  $P^L$  labeling. We will find a set of friends for  $\hat{\omega}$  with this new notion of matchability. Consider any atom  $\omega'$  in  $W_n$ . By our construction of  $W_n$  we know that there is an  $(\frac{\epsilon}{3}, N, 1 - \frac{\epsilon}{3})$  match  $\pi : B_n \rightarrow B_n$  between  $\omega$  and  $\omega'$ . Thus we know most of the indices of  $\omega$  can be divided into  $N$ -blocks so that  $\pi$  matches most of these blocks perfectly to  $N$ -blocks in  $\omega'$ .

By our construction of  $W_n$ , we know we can find  $x \in \omega$  and  $x' \in \omega'$  such that  $x, x' \in H$ . We wish to remove from consideration those matched  $N$ -blocks which, in the orbit of either  $x$  or  $x'$ , contain an occurrence of the set  $B$ . By the definition of  $H$  we know that there are less than  $2\epsilon_1 n^d$  occurrences of  $B$  in either block of the orbit; hence we will have thrown away at most  $4\epsilon_1 n^d$   $N$ -blocks. By equation (14) this is a set of indices of proportion less than  $\frac{\epsilon}{3}$  of an  $n$ -name.

Call the first lexicographic index of any  $N$ -box its base point. Consider one of the remaining  $N$ -boxes in  $\hat{\omega}$ , and suppose the  $R$ -labeling at  $\vec{v}$ , its base point, is  $i$ . We claim the following: of the extension atoms of  $\omega'$  which lie in  $\hat{W}_n$ , those which

see the  $R$ -label  $i$  at index  $\vec{\pi}(v)$  have measure at least  $\frac{\beta_1}{2}$  and at most  $\frac{\beta_2}{2}$ . To see why, note that at least  $\beta_1$  and at most  $\beta_2$  of the set of all extension atoms of  $\omega'$  have  $R$ -label  $i$  at index  $\pi(\vec{v})$ . Of the set of points in this collection of extension atoms, at most  $\beta_2\mu(\omega' \cap H^c)$  are in  $H^c \times C$ . We are considering only those extensions which are at least half covered by  $H$ , so we remove from consideration a collection of atoms of measure at most  $2 \times \beta_2\mu(\omega' \cap H^c)$ , which is less than  $2\beta_2\frac{1}{4}\frac{\beta_1}{\beta_2}\epsilon\mu(\omega')$ . This leaves us with a collection of extension atoms of measure at least  $\beta_1\mu(\omega') - \frac{\beta_1}{2}\epsilon\mu(\omega') > \frac{\beta_1}{2}\mu(\omega')$ .

So, for each such base point  $\vec{v}$  in  $\omega$ , and for any other  $\omega' \in W_n$ , at least  $\frac{\beta_1}{2}\mu(\omega')$  of the extension atoms of  $\omega'$  have  $R$ -labels at  $\pi(\vec{v})$  which agree with the  $R$ -label  $\vec{v}$  in  $\omega$ . Thus for at least  $\frac{\beta_1}{4}\mu(\omega')$  of the extensions of  $\omega'$ , the  $R$ -label must agree with  $\hat{\omega}$  at the base point of at least  $\frac{\beta_1}{4}$  of the remaining  $N$ -boxes of  $\omega$ .

Take these extensions over all atoms  $\omega' \in W_n$ , and call this set  $\hat{F}(\hat{\omega})$ . Note that by the above calculation and condition (19) we have

$$(20) \quad \mu(\hat{F}(\hat{\omega})) > \frac{\beta_1}{4}\mu(W_n) > \frac{\beta_1}{4}(1 - \epsilon) > \frac{\beta_1}{8}.$$

We now argue that for all  $\hat{\omega}' \in \hat{F}(\hat{\omega})$ ,  $\hat{\omega}$  and  $\hat{\omega}'$  are friends in the modified sense of matching. Fix  $\hat{\omega}' \in \hat{F}(\hat{\omega})$  and consider an  $N$ -box in  $\hat{\omega} \circ \pi$  for which the  $P^L$ -label agrees with the  $N$ -box at the same location in  $\hat{\omega}'$  and the  $R$ -label at  $\vec{v}$  and  $\pi(\vec{v})$  (the respective base points of the  $N$  boxes) is the same. Let  $x$  and  $x'$  be as before.

Since the  $P^L$ -labels of the  $N$ -boxes agree, the points  $x$  and  $x'$  visit the same  $P^L$  elements in those pieces of their orbit. Hence for every  $\vec{m} = (m_1, \dots, m_d) \in B_N$  and  $i = 1, \dots, d$  we have

$$(21) \quad \tilde{h}_i(T^{\vec{v}+\vec{m}}x) = \tilde{h}_i(T^{\pi(\vec{v})+\vec{m}}x').$$

Recall that we are only considering  $N$ -blocks which don't contain an occurrence of  $B$  in either orbit. Further, by our choice of  $L$  we have, for each  $\vec{m} = (m_1, \dots, m_d) \in B_N$ ,

$$(22) \quad |h(T^{\vec{v}}x, \vec{m}) - \tilde{h}_1(T^{\vec{v}}x) - \dots - \tilde{h}_1(T^{\vec{v}+m_1\vec{e}_1+\dots+m_{d-1}\vec{e}_{d-1}}x) \\ - \dots - \tilde{h}_d(T^{\vec{v}+m_1\vec{e}_1+\dots+(m_{d-1})\vec{e}_{d-1}}x)| \leq \epsilon_1 dN,$$

and the same is true for  $x'$ . So for any  $\vec{m} \in B_N$  the difference in the rotations given by  $h(x, \vec{m} + \vec{v})$  and  $h(x', \vec{m} + \pi(\vec{v}))$  cannot differ by more than  $2\epsilon_1 dN$ . By equation (14) this is less than  $d_2$ . So, if the  $R$ -label at  $\vec{v}$  and  $\pi(\vec{v})$  is the same in  $\hat{\omega}$  and  $\hat{\omega}'$ , then the  $R$ -labels at  $\vec{v} + \vec{m}$  in  $\hat{\omega}$  and  $\pi(\vec{v}) + \vec{m}$  in  $\hat{\omega}'$  must match in the modified sense. Hence, the  $R$ -label of the entire  $N$ -block in  $\hat{\omega} \circ \pi$  must match in the modified sense that in  $\hat{\omega}'$ .

Thus, in the modified sense, we have  $(\frac{\epsilon}{3}, N, \frac{\beta_1}{4}(1 - \frac{\epsilon}{3} - \frac{\epsilon}{3}))$  matchability between  $\hat{\omega}$  and  $\hat{\omega}'$ , and  $T^h$  satisfies a modified friendship condition with  $c_1 = c_2 = \frac{\beta_1}{8}$ . We can argue, as in Corollary 3.7, that this will yield a modified version of LB.

We now argue that  $(T^h, P \times Q)$  satisfies Definition 3.1.

Fix  $\epsilon > 0$ . By our above comments and a modified version of Lemma 2.5 we know we can find  $m > 0$  such that for all  $n > m$  there is a set  $\hat{W}_n$  of  $P^L \times R$  atoms with  $\hat{\mu}W_n > 1 - \frac{\epsilon}{2}$ , and all  $\hat{\omega}, \hat{\omega}' \in \hat{W}_n$  are (modified)  $\vec{f}_n$ -close enough that they are  $(\frac{\epsilon}{2}, N, 1 - \frac{\epsilon}{2})$  modified matchable.

Suppose that the partition  $R$  was constructed so that  $\beta_2$  in our above discussion was small enough that if  $\mathcal{A}$  is the collection of sets in  $R$  which either intersect the

boundary of  $Q$  or lie adjacent to a set which intersects the boundary of  $Q$ , then  $\mathcal{A}$  has measure less than  $\frac{\epsilon}{8N^d}$ . Let

$$\hat{C} = \left\{ (x, \theta) : \frac{|\vec{v} \in B_n : T_{\vec{v}}^h(x, \theta) \in X \times \mathcal{A}|}{|B_n|} < \frac{\epsilon}{4N^d} \right\},$$

and using the ergodic theorem assume  $m$  is large enough so that, for all  $n > m$ ,  $\hat{\mu}\hat{C} > 1 - \frac{\epsilon}{2}$ . Now throw away from  $\hat{W}_n$  any atom  $\hat{\omega}$  for which  $\hat{\mu}(\hat{\omega} \cap \hat{C}) = 0$ . Call the remaining set  $\hat{W}_n$ , and note that  $\hat{\mu}\hat{W}_n > 1 - \epsilon$ .

Fix  $\hat{\omega}$  and  $\hat{\omega}'$  in  $\hat{W}_n$ , and  $(x, \theta) \in \hat{\omega}$  and  $(x', \theta') \in \hat{\omega}'$  such that  $(x, \theta), (x', \theta') \in \hat{C}$ . By the definition of  $C$  we know that by throwing away at most  $\frac{\epsilon}{2N^d}|B_n|$   $N$ -boxes, we can guarantee that each modified matched  $N$ -box does not visit a set from the collection  $\mathcal{A}$  along the orbit of  $x$  and  $x'$ . Note that this eliminates a set of indices  $< \frac{\epsilon}{2}|B_n|$ .

On the remaining  $N$ -blocks it now follows that each  $R$ -set corresponds to a unique element of the partition  $Q$ . Hence we can, in a well defined manner, erase the  $R$ -labels and replace them by  $Q$  labels. Further, by equation (13) any two sets whose labels agree in the modified sense and do not belong to  $\mathcal{A}$  must lie in the same  $Q$  element. Hence, on the remaining  $N$ -boxes in  $\hat{\omega}$  and  $\hat{\omega}'$ , the modified  $R$ -label match translates to an actual  $Q$ -label match.

It follows then that the  $(P^L, Q)$  names are  $(\frac{\epsilon}{2}, N, 1 - \epsilon)$  matchable. Hence  $\bar{f}_n(\hat{\omega}, \hat{\omega}') < \epsilon$ , and we are done.  $\square$

#### REFERENCES

- [1] Hasfura-Buenaga, J. R. *The equivalence theorem for  $\mathbb{Z}^d$ -actions of positive entropy*, Ergodic Theory Dynam. Systems, (1992), **12**, 725-741. MR **94f**:28021
- [2] Johnson, A. and Şahin, A. *Rank one and loosely Bernoulli actions in  $\mathbb{Z}^d$* , Ergodic Theory Dynam. Systems, (1998), **8** no. 5, 1159-1172. CMP 99:03
- [3] Kammeyer, J. *A classification of the isometric extensions of a multidimensional Bernoulli shift*, Ergodic Theory Dynam. Systems, (1992), **12**, 267-282. MR **93i**:28015
- [4] Kammeyer, J. and Rudolph, D. *Restricted orbit equivalence for  $\mathbb{Z}^d$  actions*, Ergodic Theory Dynam. Systems, (1997), **17**, 1083-1129. MR **99c**:28057
- [5] Ornstein, D., Rudolph, D., and Weiss, B. *Equivalence of measure preserving transformations*, Mem. Amer. Math. Soc., (1982), **262**. MR **85e**:28026

DEPARTMENT OF MATHEMATICS AND STATISTICS, SWARTHMORE COLLEGE, SWARTHMORE, PENNSYLVANIA 19081

*E-mail address:* aimee@swarthmore.edu

DEPARTMENT OF MATHEMATICS, NORTH DAKOTA STATE UNIVERSITY, FARGO, NORTH DAKOTA 58103

*E-mail address:* sahin@plains.nodak.edu