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POSITIVITY OF EQUIVARIANT GROMOV-WITTEN INVARIANTS

DAVE ANDERSON AND LINDA CHEN

Abstract. We show that the equivariant Gromov-Witten invariants of a projective homogeneous space G/P exhibit Graham-positivity: when expressed as polynomials in the positive roots, they have nonnegative coefficients.

1. Introduction

Let $X = G/P$ be a projective homogeneous variety, for a complex reductive Lie group G and parabolic subgroup P. Fix a maximal torus and Borel subgroup $T \subset B \subseteq P$, and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the corresponding set of simple roots, making the roots of B positive. Let $W_P \subseteq W$ be the Weyl groups for P and G, respectively. Let B^- be the opposite Borel subgroup. The classes of the *Schubert varieties* $X(w) = \overline{BwP/P}$ and *opposite Schubert varieties* $Y(w) = \overline{B^- wP/P}$ give Poincaré dual bases of the equivariant cohomology ring H_T^*X, as w ranges over the set W^P of minimal coset representatives for W/W_P. Write $x(w) = [X(w)]^T$ and $y(w) = [Y(w)]^T$ for these classes.

A positivity property for multiplication in these bases was proved by Graham:

Theorem 1.1 ([G]). *Writing*

$$y(u) \cdot y(v) = \sum_w c_{u,v}^w \, y(w)$$

in H_T^*X, the coefficient $c_{u,v}^w$ lies in $\mathbb{N}[\alpha_1, \ldots, \alpha_n]$.

Following [K], the *equivariant Gromov-Witten invariants* are defined as follows. Let $\mathbf{d} \in H_2(X, \mathbb{Z})$ be an effective class; taking the basis of Schubert curves $x(s_\alpha)$, one can identify $\mathbf{d}$ with a tuple of nonnegative integers $(d_1, \ldots, d_k)$. Let $\overline{M} = \overline{M}_{0,r+1}(X, \mathbf{d})$ denote the Kontsevich moduli space of stable maps. This comes with $r+1$ *evaluation maps* $\mathrm{ev}_i : \overline{M} \to X$, as well as the standard map $\pi \colon \overline{M} \to \mathrm{pt}$.

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Definition 1.2. The **equivariant Gromov-Witten invariant** associated to classes $\sigma_1, \ldots, \sigma_{r+1}$ is

$$I^T_{\mathbf{d}}(\sigma_1 \cdots \sigma_{r+1}) := \pi^T_*(\mathrm{ev}_1^*\sigma_1 \cdots \mathrm{ev}_{r+1}^*\sigma_{r+1})$$

in $H^*_T(\mathrm{pt})$, where π^T_* is the equivariant pushforward $H^*_T\overline{M} \to H^*_T(\mathrm{pt})$.

When $r = 2$, these define *equivariant quantum Littlewood-Richardson (EQLR) coefficients*:

$$c^{w,\mathbf{d}}_{u,v} = I^T_{\mathbf{d}}(y(u) \cdot y(v) \cdot x(w)).$$

The EQLR coefficients were shown to be Graham-positive, in the sense of Theorem 1.1, by Mihalcea in [M]. Remarkably, they define an associative product in the *equivariant (small) quantum cohomology ring* QH^*_TX, via

$$y(u) \circ y(v) = \sum_{w,\mathbf{d}} \mathbf{q}^{\mathbf{d}}\, c^{w,\mathbf{d}}_{u,v}\, y(w),$$

so Mihalcea's result is a generalization of Graham's to the setting of equivariant quantum Schubert calculus.

In this note, we will show that the multiple-point equivariant Gromov-Witten invariants are Graham-positive:

Theorem 1.3. *For any elements $v_1, \ldots, v_r, w \in W^P$, the equivariant Gromov-Witten invariant*

$$I^T_{\mathbf{d}}(y(v_1) \cdots y(v_r) \cdot x(w))$$

lies in $\mathbb{N}[\alpha_1, \ldots, \alpha_n]$.

Associativity of the equivariant quantum ring QH^*_TX defines (generalized) EQLR coefficients $c^{w,\mathbf{d}}_{v_1,\ldots,v_r}$:

$$y(v_1) \circ \cdots \circ y(v_r) = \sum_{w,\mathbf{d}} \mathbf{q}^{\mathbf{d}}\, c^{w,\mathbf{d}}_{v_1,\ldots,v_r}\, y(w).$$

By induction using the $r = 2$ case of Theorem 1.3, it follows that these EQLR coefficients are also Graham-positive; indeed, the associativity relations are subtraction-free. This gives a new proof of Mihalcea's positivity theorem. For $r > 2$, however, the EQLR coefficients $c^{w,\mathbf{d}}_{v_1,\ldots,v_r}$ are not the same as the equivariant Gromov-Witten invariants in Theorem 1.3.

The proof of Theorem 1.3 is given in §4; the idea is to represent the coefficients of this polynomial as degrees of effective zero-cycles, using a transversality argument (Theorem 4.4). An inspection of Mihalcea's proof of positivity for EQLR coefficients suggests that his method should also work for Gromov-Witten invariants, but we find our geometric interpretation of the coefficients appealing. Moreover, we use the dimension estimates from §4 to derive a Giambelli formula for $QH^*_T(SL_n/P)$ in [AC].

Remark 1.4. As in [G], there is a corresponding positivity theorem with the roles of positive and negative roots interchanged: the Gromov-Witten invariants $I^T_{\mathbf{d}}(x(v_1) \cdots x(v_r) \cdot y(w))$ lie in $\mathbb{N}[-\alpha_1, \ldots, -\alpha_n]$. All the arguments proceed in exactly the same manner. In fact, it is this version (for $r = 2$) that is treated in [M].

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2. Setup

We assume G is an adjoint group, so that the simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ form a basis for the character group of T. We fix the basis $-\Delta = \{-\alpha_1, \ldots, -\alpha_n\}$ of *negative* simple roots, and use it to identify T with $(\mathbb{C}^*)^n$.

2.1. **Equivariant cohomology.** Let $\mathbb{E}T \to \mathbb{B}T$ be the universal principal T-bundle; that is, $\mathbb{E}T$ is a contractible space with a free right T-action, and $\mathbb{B}T = \mathbb{E}T/T$. By definition, the equivariant cohomology of a T-variety Z is the ordinary (singular) cohomology of the *Borel mixing space* $\mathbb{E}T \times^T Z$. (This notation means quotient by the relation $(e \cdot t, z) \sim (e, t \cdot z)$.) While $\mathbb{E}T$ is infinite-dimensional, it may be approximated by finite-dimensional smooth varieties. We will set $\mathbb{E} = (\mathbb{C}^m \smallsetminus \{0\})^n$, with $T = (\mathbb{C}^*)^n$ acting by scaling each factor. For fixed k and $m \gg 0$, one has natural isomorphisms

$$H_T^* Z := H^*(\mathbb{E}T \times^T Z) \cong H^*(\mathbb{E} \times^T Z),$$

so any given computation may be done with these approximation spaces.

Note that $\mathbb{B} = \mathbb{E}/T$ is isomorphic to $(\mathbb{P}^{m-1})^n$. For a T-variety Z, we will generally use calligraphic letters to denote the corresponding approximation space: $\mathcal{Z} = \mathbb{E} \times^T Z$, always understanding a suitably large fixed m. This is a fiber bundle over $\mathbb{B}$, with fiber Z.

For each $j = 0, \ldots, m-1$, we fix transverse linear subspaces $\mathbb{P}^{m-1-j}$ and $\widetilde{\mathbb{P}}^j$ inside $\mathbb{P}^{m-1}$, and for each multi-index $J = (j_1, \ldots, j_n)$ with $0 \leq j_i \leq m-1$, we set

$$\mathbb{B}_J = \widetilde{\mathbb{P}}^{j_1} \times \cdots \times \widetilde{\mathbb{P}}^{j_n} \qquad \text{and} \qquad \mathbb{B}^J = \mathbb{P}^{m-1-j_1} \times \cdots \times \mathbb{P}^{m-1-j_n}.$$

So $\dim \mathbb{B}_J = \operatorname{codim} \mathbb{B}^J = |J| := j_1 + \cdots + j_n$. Similarly, write $\mathcal{Z}_J = (\pi^T)^{-1}\mathbb{B}_J$ and $\mathcal{Z}^J = (\pi^T)^{-1}\mathbb{B}^J$, where $\pi^T : \mathcal{Z} \to \mathbb{B}$ is the projection. The notation is chosen to suggest an identification of the pushforward for this fiber bundle with the equivariant pushforward $\pi_*^T : H_T^* Z \to H_T^*(\mathrm{pt})$.

Let $\mathcal{O}_i(-1)$ be the tautological bundle on the ith factor of $\mathbb{B} = (\mathbb{P}^{m-1})^n$. The choice of basis $-\Delta$ for the character group of T yields an equality $\alpha_i = c_1(\mathcal{O}_i(1))$. If $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$ is a root, we will sometimes write $\mathcal{O}(\alpha) = \mathcal{O}_1(a_1) \otimes \cdots \otimes \mathcal{O}_n(a_n)$ for the corresponding line bundle, so $c_1(\mathcal{O}(\alpha)) = \alpha$. Note that $\mathcal{O}(\alpha)$ is globally generated if and only if α is a positive root.

From the definitions, we have

$$[\mathbb{B}^J] = \alpha^J := \alpha_1^{j_1} \cdots \alpha_n^{j_n}$$

in $H^*\mathbb{B}$. As a consequence, suppose $c = \sum_J c_J \alpha^J$ is an element of $H^*\mathbb{B} = H_T^*(\mathrm{pt})$, with $c_J \in \mathbb{Z}$. Using Poincaré duality on $\mathbb{B}$, we have $c_J = \pi_*^{\mathbb{B}}(c \cdot [\mathbb{B}_J])$, where $\pi^{\mathbb{B}}$ is the map $\mathbb{B} \to \mathrm{pt}$.

When $c = \pi_*^T(\sigma)$ comes from a class $\sigma \in H_T^* Z = H^*\mathcal{Z}$ for a complete T-variety Z, we have

$$(*) \qquad c_J = \pi_*^{\mathcal{Z}}(\sigma \cdot [\mathcal{Z}_J]),$$

using the projection formula and the fact that $(\pi^T)^*[\mathbb{B}_J] = [\mathcal{Z}_J]$. (The latter holds since $\pi^T : \mathcal{Z} \to \mathbb{B}$ is flat; for a more general argument in the case where Z is Cohen-Macaulay, see [FPr, Lemma, p. 108].)

2.2. **Stable maps.** We briefly summarize some basic facts about the space of stable maps; proofs and details may be found in [FPa]. As always, $X = G/P$. The (coarse) moduli space $\overline{M} = \overline{M}_{0,r+1}(X, \mathbf{d})$ parametrizes data $(f, C, p_1, \ldots, p_{r+1})$, where C is a connected nodal curve of genus 0, and $f : C \to X$ is a map with $f_*[C] = \mathbf{d}$ in $H_2(X, \mathbb{Z})$. (Stability means that any irreducible component of C which is collapsed by f has at least three "special" points, i.e., marked points p_i or nodes.)

The space of stable maps is an irreducible projective variety of dimension

$$\dim \overline{M} = \dim X + \langle c_1(TX), \mathbf{d} \rangle + r - 2,$$

and has quotient singularities, and therefore rational singularities; in particular, it is Cohen-Macaulay. The locus parametrizing maps with irreducible domain is a dense open subset $M = M_{0,r+1}(X, \mathbf{d}) \subseteq \overline{M}$, and the complement is a divisor $\partial \overline{M} = \overline{M} \smallsetminus M$.

There are natural *evaluation maps* $\mathrm{ev}_i : \overline{M} \to X$, defined by sending a stable map $(f, C, p_1, \ldots, p_{r+1})$ to $f(p_i)$. The group G acts on $\overline{M}$ by $g \cdot (f, C, \{p_i\}) = (g \cdot f, C, \{p_i\})$, and the evaluation maps are equivariant for the actions of G on $\overline{M}$ and X. Considering the induced action of $T \subset G$, we obtain maps $\mathrm{ev}_i^T : \overline{\mathcal{M}} \to \mathcal{X}$ on Borel mixing spaces, which commute with the projections to $\mathbb{B}$.

Remark 2.1. The significance of $\overline{M}$ being Cohen-Macaulay is that the usual apparatus of intersection theory applies; see especially Lemma 4.2 below. In fact, the corresponding moduli stack is smooth, so one could argue directly using intersection theory on stacks.

3. A group action

In [A] and [AGM], a large group action on the mixing space $\mathcal{X}$ was constructed; we describe it here. The idea is to mix the transitive action of $(PGL_m)^n$ on $\mathbb{B}$ with a "fiberwise" action by Borel groups. Let T act on G by conjugation, and let $\mathcal{G} = \mathbb{E} \times^T G$ be the corresponding group scheme over $\mathbb{B}$. Because T acts by conjugation, the evident action $(\mathbb{E} \times G) \times_{\mathbb{E}} (\mathbb{E} \times X) \to \mathbb{E} \times X$ descends to an action $\mathcal{G} \times_{\mathbb{B}} \mathcal{X} \to \mathcal{X}$.

Let $U \subset B \subset G$ be the unipotent radical of B, and let $\mathcal{U} \subset \mathcal{B} \subset \mathcal{G}$ be the corresponding group bundles over $\mathbb{B}$. As a variety, $\mathcal{U}$ is isomorphic to the vector bundle $\bigoplus_{\alpha\in R^+}\mathcal{O}(\alpha)$ on $\mathbb{B}$, where the sum is over the positive roots. Now consider the group of sections $\Gamma_0 = \mathrm{Hom}_{\mathbb{B}}(\mathbb{B},\mathcal{U})$; this is a connected algebraic group over $\mathbb{C}$. As observed in §2.1, each $\mathcal{O}(\alpha)$ is globally generated. It follows that for each $x \in \mathbb{B}$, the map $\Gamma_0 \to \mathcal{U}_x$ given by evaluating sections at x is surjective, and therefore we have:

Lemma 3.1 ([AGM, Lemma 6.3])**.** *Let Γ be the* ***mixing group*** *$\Gamma_0 \rtimes (PGL_m)^n$, where $(PGL_m)^n$ acts on Γ_0 via its action on $\mathbb{B}$. Then Γ is a connected linear algebraic group acting on $\mathcal{X}$, with (finitely many) orbits whose closures are the Schubert bundles $\mathcal{X}(w)$.*

Similarly, the group $\Gamma^{(r)} = \Gamma_0^r \rtimes (PGL_m)^n$ acts on the r-fold fiber product $\mathcal{X}\times_{\mathbb{B}}\cdots\times_{\mathbb{B}}\mathcal{X}$, with orbit closures $\mathcal{X}(w_1)\times_{\mathbb{B}}\cdots\times_{\mathbb{B}}\mathcal{X}(w_r)$.

4. Transverality

A map $f : Y \to X$ is said to be **dimensionally transverse** to a subvariety $W \subseteq X$ if $\mathrm{codim}_Y(f^{-1}W) = \mathrm{codim}_X(W)$. We will need the following version of Kleiman's transversality theorem; see [Kl] and [S]. As a matter of notation, if a group Γ acts on X, we write $\gamma f : \gamma Y \to X$ for the composition $Y \xrightarrow{f} X \xrightarrow{\cdot\gamma} X$, i.e., the translation of f by the action of $\gamma \in \Gamma$.

Proposition 4.1. *Let Γ be a group acting on a smooth variety X, and suppose $f : Y \to X$ is dimensionally transverse to the orbits of Γ. Assume Y is Cohen-Macaulay. Let $g : Z \to X$ be any map. Then for a general element $\gamma \in \Gamma$, the fiber product $V_\gamma = \gamma Y \times_X Z$ has dimension equal to $\dim Y + \dim Z - \dim X$.*

The essential point in the proof is that the hypotheses imply the map $\Gamma \times Y \to X$ is flat.

We will also use the following lemma:

Lemma 4.2 ([FPr, Lemma, p. 108])**.** *Let $f : Z \to X$ be a morphism from a pure-dimensional Cohen-Macaulay scheme Z to a nonsingular variety X, and let $W \subseteq X$ be a closed Cohen-Macaulay subscheme of pure codimension d. Let $V = f^{-1}W$, and assume* $\mathrm{codim}_Z(V) = d$*. Then V is Cohen-Macaulay, and $f^*[W] = [V]$.*

Now resume the previous notation, so $X = G/P$ and $\overline{M} = \overline{M}_{0,r+1}(X,\mathbf{d})$. Since each evaluation map $\mathrm{ev}_i : \overline{M} \to X$ is G-equivariant, it is flat. If $W \subseteq X$ is any Cohen-Macaulay subscheme of codimension d, it follows that $\mathrm{ev}_i^{-1}W \subseteq \overline{M}$ has the same properties, and similarly, $(\mathrm{ev}_i^T)^{-1}\mathcal{W} \subseteq \overline{\mathcal{M}}$. In particular, the subscheme

$$\mathcal{Z} = (\mathrm{ev}_{r+1}^T)^{-1}(\mathcal{X}(w)) \subseteq \overline{\mathcal{M}}$$

is Cohen-Macaulay of codimension $\dim X - \ell(w)$, and we have $[\mathcal{Z}] = (\mathrm{ev}_{r+1}^T)^*(x(w))$ by Lemma 4.2. Similarly, we have

$$[\mathcal{Z}_J] = (\mathrm{ev}_{r+1}^T)^*(x(w)) \cdot [\overline{\mathcal{M}}_J] \tag{†}$$

Consider the map $\mathrm{ev} = \mathrm{ev}_1 \times \cdots \times \mathrm{ev}_r : \overline{M} \to X^r$ and the corresponding map on mixing spaces $\mathrm{ev}^T : \overline{\mathcal{M}} \to \mathcal{X}^r$. Let $\mathcal{Y} = \mathcal{Y}(v_1) \times_{\mathbb{B}} \times \cdots \times_{\mathbb{B}} \mathcal{Y}(v_r)$, and let f be the inclusion of $\mathcal{Y}$ in the r-fold fiber product $\mathcal{X}^r$.

Lemma 4.3. *Let $\gamma = (\gamma_1, \ldots, \gamma_r)$ be a general element in $\Gamma^{(r)}$.*

(a) *The intersection*

$$\begin{aligned} V_\gamma &= (\mathrm{ev}_1^T)^{-1}(\gamma_1 \mathcal{Y}(v_1)) \cap \cdots \cap (\mathrm{ev}_r^T)^{-1}(\gamma_r \mathcal{Y}(v_r)) \cap \mathcal{Z}_J \\ &= \gamma\mathcal{Y} \times_{\mathcal{X}^r} \mathcal{Z}_J \end{aligned}$$

is Cohen-Macaulay and pure-dimensional, of dimension $\dim \overline{M} + |J| - \dim X + \ell(w) - \ell(v_1) - \cdots - \ell(v_r)$. *(In the fiber product, $\mathcal{Z}_J$ maps to $\mathcal{X}^r$ by the restriction of* ev^T*.)*

(b) *Similarly, the intersection*

$$\begin{aligned} \partial V_\gamma &= (\mathrm{ev}_1^T)^{-1}(\gamma_1 \mathcal{Y}(v_1)) \cap \cdots \cap (\mathrm{ev}_r^T)^{-1}(\gamma_r \mathcal{Y}(v_r)) \cap \mathcal{Z}_J \cap \partial\overline{\mathcal{M}} \\ &= \gamma\mathcal{Y} \times_{\mathcal{X}^r} (\mathcal{Z}_J \cap \partial\overline{\mathcal{M}}) \end{aligned}$$

has pure dimension $\dim \overline{M} + |J| - \dim X + \ell(w) - \ell(v_1) - \cdots - \ell(v_r) - 1$.

In particular, when $\dim \overline{M} + |J| - \dim X + \ell(w) - \ell(v_1) - \cdots - \ell(v_r) = 0$, *the intersection V_γ consists of finitely many points contained in $\mathcal{M}$.*

Proof. Note that $\mathcal{Z}_J$ is Cohen-Macaulay (since $\mathcal{Z}$ is), of dimension $\dim \overline{M} + |J| - \dim X + \ell(w)$. Each opposite Schubert bundle $\mathcal{Y}(v)$ intersects each Γ-orbit closure $\mathcal{X}(w)$ properly, so the map $f : \mathcal{Y} \hookrightarrow \mathcal{X}^r$ is dimensionally transverse to the $\Gamma^{(r)}$-orbits. The first statement follows by an application of Proposition 4.1.

The second statement is proved similarly; note that the divisor $\partial\overline{M}$ is Cohen-Macaulay and G-invariant, and the same argument as before shows that $\mathcal{Z}_J \cap \partial\overline{\mathcal{M}}$ is a Cohen-Macaulay divisor in $\mathcal{Z}_J$. $\square$

We can now prove Theorem 1.3. In fact, it follows immediately from $(*)$, together with a more precise statement.

Theorem 4.4. *Write $I^T_{\mathbf{d}}(y(v_1) \cdots y(v_r) \cdot x(w)) = \sum c_J \alpha^J$ in $H^*_T(\mathrm{pt})$. Then, with notation as in Lemma 4.3, we have*

$$c_J = \deg(V_\gamma)$$

when $\dim \overline{M} + |J| - \dim X + \ell(w) - \ell(v_1) - \cdots - \ell(v_r) = 0$, *and $c_J = 0$ otherwise.*

In particular, since V_γ is an effective cycle, c_J is a nonnegative integer.

Proof. Using $(*)$ from §2.1, we have

$$c_J = \pi_*^{\overline{\mathcal{M}}}((\mathrm{ev}_1^T)^* y(v_1) \cdots (\mathrm{ev}_r^T)^* y(v_r) \cdot (\mathrm{ev}_{r+1}^T)^* x(w) \cdot [\overline{\mathcal{M}}_J]).$$

The claim is that $(\mathrm{ev}_1^T)^* y(v_1) \cdots (\mathrm{ev}_r^T)^* y(v_r) \cdot (\mathrm{ev}_{r+1}^T)^* x(w) \cdot [\overline{\mathcal{M}}_J] = [V_\gamma]$ in $H^*\overline{\mathcal{M}}$.

First observe that $(\mathrm{ev}_1^T)^* y(v_1)\cdots(\mathrm{ev}_r^T)^* y(v_r) = (\mathrm{ev}^T)^*(y(v_1)\times\cdots\times y(v_r))$. Since $\Gamma^{(r)}$ is connected, we have $[\gamma\mathcal{Y}] = [\mathcal{Y}] = y(v_1)\times\cdots\times y(r)$ in $H^*(\mathcal{X}^r) = H_T^*(X^r)$. By the same argument as in the paragraph after Lemma 4.2, we have $[(\mathrm{ev}^T)^{-1}(\gamma\mathcal{Y})] = (\mathrm{ev}^T)^*(y(v_1)\times\cdots\times y(v_r))$.

By (†), we have $[\mathcal{Z}_J] = (\mathrm{ev}_{r+1}^T)^* x(w)\cdot[\overline{\mathcal{M}}_J]$. Since $(\mathrm{ev}^T)^{-1}(\gamma\mathcal{Y})$ and $\mathcal{Z}_J$ intersect properly in V_γ by Lemma 4.3, we have $[(\mathrm{ev}^T)^{-1}(\gamma\mathcal{Y})]\cdot[\mathcal{Z}_J] = [V_\gamma]$, as desired. $\square$

Remark 4.5. Let $\overline{M}_{0,r+1}$ be the moduli space of stable curves with $r+1$ marked points; this is a nonsingular projective variety of dimension $r-2$. Since T acts trivially on this space, the corresponding mixing space is $\overline{\mathcal{M}}_{0,r+1} = \mathbb{B}\times\overline{M}_{0,r+1}$. The forgetful map $\varphi:\overline{M}\to\overline{M}_{0,r+1}$ induces a map $\overline{\mathcal{M}}\to\overline{\mathcal{M}}_{0,r+1}$. Let $\widetilde{\varphi}:\overline{\mathcal{M}}\to\overline{M}_{0,r+1}$ be the composition with the second projection, and for $x\in\overline{M}_{0,r+1}$, write $\overline{\mathcal{M}}(x) = \widetilde{\varphi}^{-1}(x)$. Using the notation of Lemma 4.3, the same arguments used in the proof of the lemma also establish the following dimension counts:

(a) Let $V_\gamma(x) = V_\gamma\cap\overline{\mathcal{M}}(x)$. Then $V_\gamma(x)$ is Cohen-Macaulay, of pure dimension $\dim\overline{M}+|J|-(\dim X-\ell(w))-\ell(v_1)-\cdots-\ell(v_r)-(r-2)$.

(b) Let $\partial V_\gamma(x) = \partial V_\gamma\cap\overline{\mathcal{M}}(x)$. Then $\partial V_\gamma(x)$ is Cohen-Macaulay, of pure dimension $\dim\overline{M}+|J|-(\dim X-\ell(w))-\ell(v_1)-\cdots-\ell(v_r)-(r-2)-1$.

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