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# GENERALIZED COKÄHLER GEOMETRY AND AN APPLICATION TO GENERALIZED KÄHLER STRUCTURES

RALPH R. GOMEZ AND JANET TALVACCHIA

ABSTRACT. In this paper, we propose a generalization of classical coKähler geometry from the point of view of generalized contact metric geometry. This allows us to generalize a theorem of M.Capursi and I.Goldberg ([7],[10]) and show that the product  $M_1 \times M_2$  of generalized contact metric manifolds  $(M_i, \Phi_i, E_{\pm,i}, G_i)$ ,  $i = 1, 2$ , where  $M_1 \times M_2$  is endowed with the product (twisted) generalized complex structure induced from  $\Phi_1$  and  $\Phi_2$ , is (twisted) generalized Kähler if and only if  $(M_i, \Phi_i, E_{\pm,i}, G_i)$ ,  $i = 1, 2$  are (twisted) generalized coKähler structures. As an application of our theorem we construct new examples of twisted generalized Kähler structures on manifolds that do not admit a classical Kähler structure and we give examples of twisted generalized coKähler structures on manifolds which do not admit a classical coKähler structure.

## 1. INTRODUCTION

Consider the product of two manifolds  $M_1 \times M_2$  where each manifold has an almost contact structure  $(\phi_i, \xi_i, \eta_i)$  on  $M_i$ ,  $\phi_i$  is an endomorphism of  $TM_i$ ,  $\xi_i$  is a vector field and  $\eta_i$  is a 1-form such that  $\eta_i(\xi_i) = 1$ ,  $i = 1, 2$ . A natural question to ask is what further conditions are needed on  $M_i$  to ensure the product  $M_1 \times M_2$  is a complex manifold. Morimoto answered this question in [21]. He constructed a natural almost complex structure  $J$  on the product given by

$$(1.1) \quad J(X, Y) = (\phi_1(X) - \eta_2(Y)\xi_1, \phi_2(Y) + \eta_1(X)\xi_2)$$

where  $X \in TM_1$ ,  $Y \in TM_2$ , and showed that  $J$  is integrable if and only if each factor manifold  $M_i$  is normal almost contact. An interesting corollary of this is that the product of two odd-dimensional spheres is complex. One can further ask under what conditions on  $M_i$  the product manifold is Kähler. It is tempting to speculate that the answer should be that each factor manifold  $M_i$  is Sasakian. But that cannot be the case since this would mean the product of two Sasakian spheres would be Kähler violating Calabi and Eckmann's result that the product of odd dimensional spheres is complex and nonKähler [5]. It was Goldberg [10] and later Capursi [7] who realized the right geometric structure on the factor manifolds  $M_i$  should be a coKähler structure. A coKähler manifold is a normal almost contact metric manifold  $(\phi, \xi, \eta, g)$  where  $g$  is a Riemannian metric compatible with the other structure tensors such that both the one-form  $\eta$  is closed and the fundamental two-form  $\Omega = g(X, \phi Y)$  is closed, for any sections  $X, Y$  of  $TM$ . (See section 2 for precise definitions.) We state the theorem of Capursi and Goldberg here since the principal aim of this paper is to generalize their theorem:

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**Theorem 1.1** (Capursi,Goldberg): *Let  $(M_i, \phi_i, \xi_i, \eta_i, g_i)$ ,  $i = 1, 2$ , be almost contact metric manifolds and let  $J$  be defined as above. The manifold  $(M_1 \times M_2, J, G)$  is Kähler if and only if  $(M_i, \phi_i, \xi_i, \eta_i, g_i)$  is coKähler, where  $G = g_1 + g_2$ .*

The beginnings of coKähler geometry started in its geometric role in time-dependant mechanics (see for example [2]) and since then the subject has grown in spurts. Various results have emerged over the years, elucidating both the differential geometry and the topology of the manifolds admitting such structures. The reader should refer to [6] for a recent and comprehensive survey of coKähler geometry, and more generally, cosymplectic geometry.

The notion of a generalized complex structure and its twisted counterparts, introduced by Hitchin in his paper [17] and developed by Gualtieri ([14],[15]) is a framework that unifies both complex and symplectic structures. These structures exist only on even dimensional manifolds. The odd dimensional analog of this structure, a generalized contact structure, was taken up by Vaisman ([25],[26]), Poon, Wade [22], and Sekiya [24]. This framework unifies almost contact, contact, and cosymplectic structures. Generalized Kähler structures were introduced by Gualtieri [14],[15],[16] and have already found their way into the physics literature ([18], [19], [20], [13]).

In order to prove a generalized contact version of Theorem 1.1, the first step is to reformulate Morimoto's theorem discussed above in the generalized contact setting. This step was accomplished by the authors in [11]. Consider the generalized almost contact structure  $(\Phi_i, E_{\pm,i})$  where  $\Phi_i$  is an endomorphism of  $TM_i \oplus T^*M_i$  and  $E_{\pm,i}$  are sections of  $TM_i \oplus T^*M_i$ ,  $i = 1, 2$ , such that the conditions given in (2.3)-(2.5) are satisfied. It was shown by the authors that one can generalize equation (1.1) and this generalized almost complex structure on  $M_1 \times M_2$  is given by:

$$\begin{aligned} \mathcal{J}(X_1 + \alpha_1, X_2 + \alpha_2) = & (\Phi_1(X_1 + \alpha_1) - 2\langle E_{+,2}, X_2 + \alpha_2 \rangle E_{+,1} - 2\langle E_{-,2}, X_2 + \alpha_2 \rangle E_{-,1}, \\ (1.2) \quad & \Phi_2(X_2 + \alpha_2) + 2\langle E_{+,1}, X_1 + \alpha_1 \rangle E_{+,2} + 2\langle E_{-,1}, X_1 + \alpha_1 \rangle E_{-,2}) \end{aligned}$$

for any sections  $X_i + \alpha_i$  of  $TM_i \oplus T^*M_i$ . This formula was then used to give a proof of a generalization almost contact version of Morimoto's theorem mentioned above.

**Theorem 1.2:** [11] *Let  $M_1$  and  $M_2$  be odd dimensional smooth manifolds each with generalized almost contact structures  $(\Phi_i, E_{\pm,i})$   $i = 1, 2$ . Then  $M_1 \times M_2$  admits a generalized almost complex structure  $\mathcal{J}$ . Further  $\mathcal{J}$  is a generalized complex structure if and only if both  $(\Phi_i, E_{\pm,i})$   $i = 1, 2$  are strong generalized contact structures and  $[[E_{\pm,i}, E_{\mp,i}]] = 0$ .*

In this article, we will first propose a generalization of coKähler geometry using the language of generalized contact metric geometry and then we prove the following generalization of Theorem 1.1:

**Theorem 1.3:** *Let  $M_1$  and  $M_2$  be odd dimensional smooth manifolds each with a (twisted) generalized contact metric structure  $(\Phi, E_{\pm,i}, G_i)$ ,  $i = 1, 2$  such that on the product  $M_1 \times M_2$  are two (twisted) generalized almost complex structures:  $\mathcal{J}_1$  which is the natural generalized almost complex structure induced from  $\Phi_1$  and  $\Phi_2$  and  $\mathcal{J}_2 = G\mathcal{J}_1$  where  $G = G_1 \times G_2$ . Then  $(M_1 \times M_2, \mathcal{J}_1, \mathcal{J}_2)$  is (twisted) generalized Kähler if and only if  $(\Phi_i, E_{\pm,i}, G_i)$ ,  $i = 1, 2$  are (twisted) generalized coKähler structures.*

In section 2 we gather the basics of coKähler geometry, generalized complex, generalized Kähler, and generalized contact geometry. In section 3, we state and prove some basic properties of generalized almost contact metric structures and define the notion of a generalized coKähler structure. Then, in section 4 we prove Theorem 1.3. In section 5, we give numerous examples of generalized coKähler structures. In particular, we are able to construct almost Kähler nonKähler manifolds that admit twisted generalized Kähler structures and we are able to construct almost coKähler noncoKähler manifolds that admit twisted generalized coKähler structures.

## 2. PRELIMINARIES

In this section we will record some of the fundamental geometric structures needed for the generalization of coKähler geometry. But first, it will be worthwhile to recall the formal definition of a coKähler structure on a manifold.

An almost contact metric structure on  $M$  is given by the the following structure tensors  $(\phi, \xi, \eta, g)$  where  $\phi$  is a  $(1, 1)$  tensor field,  $\xi$  is a vector field and  $\eta$  is a 1-form, given by the following conditions

$$(2.1) \quad \phi^2 = -I + \eta \otimes \eta, \quad \eta(\xi) = 1$$

and where  $g$  is a Riemannian metric subject to the following compatibility condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for an vector fields  $X, Y \in \Gamma(TM)$ . We can use the Riemannian metric  $g$  and  $\phi$  to construct the fundamental 2-form

$$(2.2) \quad \Omega(X, Y) = g(X, \phi Y).$$

Finally, recall the Nijenhuis torsion tensor

$$N_I(X, Y) = [IX, IY] + I^2[X, Y] - I[X, IY] - I[IX, Y]$$

is defined for any  $(1, 1)$  tensor field  $I$ . An almost contact (metric) structure is normal if  $N_\phi = -2\xi \otimes d\eta$  [6]. Equivalently, an almost contact (metric) structure on  $M$  is normal if the associated almost complex structure coming from (1.1) on  $M \times \mathbb{R}$  is integrable. We are now ready for the definition of a coKähler structure.

**Definition 2.1:** *An almost coKähler manifold is an almost contact metric manifold  $(M, \phi, \xi, \eta, g)$  such that the fundamental 2-form  $\Omega$  and the 1-form  $\eta$  are closed. Furthermore,  $M$  is a coKähler manifold if the underlying almost contact structure is normal, that is  $N_\phi = -2\xi \otimes d\eta = 0$ .*

We give here some examples coKähler manifolds which will be useful in section 5.

**EXAMPLE 2.2:** On  $S^1$  we can define  $(\phi, \xi, \eta, g)$  where  $\phi = 0$ ,  $\xi = \partial_t$  where  $t$  is the coordinate on  $S^1$ ,  $\eta = dt$ , and  $g = dt \otimes dt$ . It is clear that  $S^1$  is almost coKähler. The normality follows at once since  $\phi = 0$  and  $\eta$  is closed. The manifold  $S^1$  together with this coKähler structure will be called the trivial coKähler structure on  $S^1$ .

**EXAMPLE 2.3:** [6] Let  $(N, J, g)$  be an almost Kähler manifold and form the product, denoted by  $M$ , with  $\mathbb{R}$  (or  $S^1$ ). Let  $t$  denote the coordinate on  $\mathbb{R}$  and let  $(X, f\partial_t)$  be a vector field on  $M$ , where  $f$  is any smooth function on the product. Now define an endomorphism of  $TM$  by letting  $\phi(X, f\partial_t) := (JX, 0)$ . Moreover, define  $\xi := \partial_t$  and a 1-form  $\eta = dt$ . The metric on  $M$  can be taken to be the product metric  $h = g + dt^2$ . A

straightforward calculation shows that  $(M, \phi, \xi, \eta, h)$  is coKähler if and only if  $(N, J, g)$  is Kähler.

Just as a Kähler structure imposes topological restrictions on the manifold, a coKähler structure on an odd dimensional manifold imposes some topological restrictions as well. For example, it was shown in [4] that all the Betti numbers of a compact coKähler manifold are non-zero. Here we state another topological result that we will use in the last section when we construct examples.

**Theorem 2.4:** [8] *Let  $M$  be a compact coKähler manifold. Then the first Betti number of  $M$  is odd.*

We now move to a very brief review of generalized geometric structures. Throughout this paper we let  $M$  be a smooth manifold. Consider the big tangent bundle  $TM \oplus T^*M$ . We define a neutral metric on  $TM \oplus T^*M$  by

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\beta(X) + \alpha(Y))$$

and the (H-twisted) Courant bracket by

$$[[X + \alpha, Y + \beta]]_H = [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2}d(\iota_X \beta - \iota_Y \alpha) + \iota_Y \iota_X H$$

where  $X, Y \in TM$  and  $\alpha, \beta \in T^*M$  and  $H$  is a real closed 3-form. A subbundle of  $TM \oplus T^*M$  is said to be involutive if its sections are closed under the ( $H$ -twisted) Courant bracket[14].

**Definition 2.5:** *A generalized almost complex structure on  $M$  is an endomorphism  $\mathcal{J}$  of  $TM \oplus T^*M$  such that  $\mathcal{J} + \mathcal{J}^* = 0$  and  $\mathcal{J}^2 = -Id$ . If the  $\sqrt{-1}$  eigenbundle  $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$  associated to  $\mathcal{J}$  is involutive with respect to the ( $H$ -twisted) Courant bracket, then  $\mathcal{J}$  is called an ( $H$ -twisted) generalized complex structure.*

Here are the prototypical examples in the case when  $H = 0$ :

**EXAMPLE 2.6:** [14] Let  $(M^{2n}, J)$  be a complex structure. Then we get a generalized complex structure by setting

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

**EXAMPLE 2.7:** [14] Let  $(M^{2n}, \omega)$  be a symplectic structure. Then we get a generalized complex structure by setting

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Diffeomorphisms of  $M$  preserve the Lie bracket of smooth vector fields and in fact such diffeomorphisms are the only automorphisms of the tangent bundle. But in generalized geometry, there is actually more flexibility. That is to say, given  $T \oplus T^*$  equipped with the (twisted) Courant bracket, the automorphism group is comprised of the diffeomorphisms of  $M$  and some additional symmetries called  $B$ -field transformations [14].

**Definition 2.8:** [14] *Let  $B$  be a closed two-form which we view as a map from  $T \rightarrow T^*$  given by interior product. Then the invertible bundle map*

$$e^B := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : X + \xi \mapsto X + \xi + \iota_X B$$

is called a  $B$ -field transformation.

A  $B$ -field transformation of a (twisted) generalized (almost) complex structure  $(M, e^B \mathcal{J} e^{-B})$  is again a (twisted) generalized (almost) complex structure.

Recall that we can reduce the structure group of  $T \oplus T^*$  from  $O(2n, 2n)$  to the maximal compact subgroup  $O(2n) \times O(2n)$ . This is equivalent to an orthogonal splitting of  $T \oplus T^* = V_+ \oplus V_-$ , where  $V_+$  and  $V_-$  are positive and negative definite respectfully with respect to the inner product. Thus we can define a positive definite Riemannian metric on the big tangent bundle by

$$G = \langle, \rangle|_{V_+} - \langle, \rangle|_{V_-}.$$

A positive definite metric  $G$  on  $M$  is an automorphism of  $TM \oplus T^*M$  such that  $G^* = G$  and  $G^2 = 1$ . In the presence of a generalized almost complex structure  $\mathcal{J}_1$ , if  $G$  commutes with  $\mathcal{J}_1$  ( $G\mathcal{J}_1 = \mathcal{J}_1G$ ) then  $G\mathcal{J}_1$  squares to  $-1$  and we generate a second generalized almost complex structure,  $\mathcal{J}_2 = G\mathcal{J}_1$ , such that  $\mathcal{J}_1$  and  $\mathcal{J}_2$  commute and  $G = -\mathcal{J}_1\mathcal{J}_2$ . We are now able to recall the following:

**Definition 2.9:** [14] *An ( $H$ -twisted) generalized Kähler structure is a pair of commuting ( $H$ -twisted) generalized complex structures  $\mathcal{J}_1, \mathcal{J}_2$  such that  $G = -\mathcal{J}_1\mathcal{J}_2$  is a positive definite metric on  $T \oplus T^*$ .*

The two examples just given together give the standard example of a generalized Kähler manifold in the case  $H = 0$  [14].

EXAMPLE 2.10: Consider a Kähler structure  $(\omega, J, g)$  on  $M$ . By defining  $\mathcal{J}_J$  and  $\mathcal{J}_\omega$  as in Examples 2.6 and 2.7, we obtain a generalized Kähler structure on  $M$ , where

$$G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}.$$

Let us now recall the odd dimensional analog of generalized complex geometry. We use the definition given in [24].

**Definition 2.11:** *A generalized almost contact structure on  $M$  is a triple  $(\Phi, E_\pm)$  where  $\Phi$  is an endomorphism of  $TM \oplus T^*M$ , and  $E_+$  and  $E_-$  are sections of  $TM \oplus T^*M$  which satisfy*

$$(2.3) \quad \Phi + \Phi^* = 0$$

$$(2.4) \quad \Phi \circ \Phi = -Id + E_+ \otimes E_- + E_- \otimes E_+$$

$$(2.5) \quad \langle E_\pm, E_\pm \rangle = 0, \quad 2\langle E_+, E_- \rangle = 1.$$

An easy and immediate consequence [24] of these definitions is

$$(2.6) \quad \Phi(E_\pm) = 0.$$

Now, since  $\Phi$  satisfies  $\Phi^3 + \Phi = 0$ , we see that  $\Phi$  has 0 as well as  $\pm\sqrt{-1}$  eigenvalues when viewed as an endomorphism of the complexified big tangent bundle  $(TM \oplus T^*M) \otimes \mathbb{C}$ . The kernel of  $\Phi$  is  $L_{E_+} \oplus L_{E_-}$  where  $L_{E_\pm}$  is the line bundle spanned by  $E_\pm$ . Let  $E^{(1,0)}$  be the  $\sqrt{-1}$  eigenbundle of  $\Phi$ . Let  $E^{(0,1)}$  be the  $-\sqrt{-1}$  eigenbundle. Observe:

$$E^{(1,0)} = \{X + \alpha - \sqrt{-1}\Phi(X + \alpha) \mid \langle E_\pm, X + \alpha \rangle = 0\}$$

$$E^{(0,1)} = \{X + \alpha + \sqrt{-1}\Phi(X + \alpha) | \langle E_{\pm}, X + \alpha \rangle = 0\}.$$

Then the complex line bundles

$$L^+ = L_{E_+} \oplus E^{(1,0)}$$

and

$$L^- = L_{E_-} \oplus E^{(1,0)}$$

are maximal isotropics.

**Definition 2.12:** *A generalized almost contact structure  $(\Phi, E_{\pm})$  is an  $(H$ -twisted) generalized contact structure if either  $L^+$  or  $L^-$  is closed with respect to the  $(H$ -twisted) Courant bracket. The  $(H$ -twisted) generalized contact structure is strong if both  $L^+$  and  $L^-$  are closed with respect to the  $(H$ -twisted) Courant bracket.*

Here are the standard examples in the untwisted case  $H = 0$  :

EXAMPLE 2.13: [22] Let  $(\phi, \xi, \eta)$  be a normal almost contact structure on a manifold  $M^{2n+1}$ . Then we get a generalized almost contact structure by setting

$$\Phi = \begin{pmatrix} \phi & 0 \\ 0 & -\phi^* \end{pmatrix}, \quad E_+ = \xi, \quad E_- = \eta$$

where  $(\phi^*\alpha)(X) = \alpha(\phi(X))$ ,  $X \in TM$ ,  $\alpha \in T^*M$ . Moreover,  $(\Phi, E_{\pm})$  is an example of a strong generalized almost contact structure.

EXAMPLE 2.14: [22] Let  $(M^{2n+1}, \eta)$  be a contact manifold with  $\xi$  the corresponding Reeb vector field so that

$$\iota_{\xi}d\eta = 0 \quad \eta(\xi) = 1.$$

Then

$$\rho(X) := \iota_X d\eta - \eta(X)\eta$$

is an isomorphism from the tangent bundle to the cotangent bundle. Define a bivector field by

$$\pi(\alpha, \beta) := d\eta(\rho^{-1}(\alpha), \rho^{-1}(\beta)),$$

where  $\alpha, \beta \in T^*$ . We obtain a generalized almost contact structure by setting

$$\Phi = \begin{pmatrix} 0 & \pi \\ d\eta & 0 \end{pmatrix}, \quad E_+ = \eta, \quad E_- = \xi.$$

In fact,  $(\Phi, E_{\pm})$  is an example which is not strong.

### 3. THE DEFINITION OF GENERALIZED COKÄHLER AND SOME PROPERTIES AND EXAMPLES

The classical notions of normal almost contact structures, contact metric structures, and cosymplectic structures all have analogs in the generalized context. Until now, the notion of a generalized coKähler structure has not been defined. In this section we propose a definition of a generalized coKähler structure.

Recall an almost contact structure  $(\phi, \xi, \eta)$  on an odd dimensional manifold  $M$  is called normal if the associated almost complex structure on  $M \times \mathbb{R}$  is integrable. In our previous paper [11] (see also the introduction), we proved that the product of generalized almost contact structures  $(M_i, \Phi_i, E_{\pm, i})$ ,  $i = 1, 2$  is a generalized complex

structure if and only if each  $\Phi_i$  is strong and  $[[E_{\pm,i}, E_{\mp,i}]] = 0$ . Thus, in keeping with the classical notion of normal almost contact structure, we have

**Definition 3.1:** A generalized almost contact structure  $(M, \Phi, E_{\pm})$  is a normal generalized contact structure if  $\Phi$  is strong with respect to the (H-twisted) Courant bracket and  $[[E_+, E_-]]_H = 0$ .

As a consequence of the theorem in our previous paper [11] if we take  $M$  to have a normal generalized contact structure and for  $\mathbb{R}$  to have the trivial normal generalized contact structure ( $\Phi = 0, E_+ = dt, E_- = \partial t$ ), then the cone  $M \times \mathbb{R}$  admits a generalized complex structure. Thus our definition is consistent with the more restrictive definition of a normal generalized almost contact structure given in [25].

An almost contact metric structure on  $M^{2n+1}$  is an almost contact structure  $(\phi, \xi, \eta)$  and a Riemannian metric  $g$  that satisfies  $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ . Sekiya [24] defined a generalized almost contact metric structure as a generalized almost contact structure  $(\Phi, E_{\pm})$  along with a generalized Riemannian metric  $G$  that satisfies

$$(3.1) \quad -\Phi G \Phi = G - E_+ \otimes E_+ - E_- \otimes E_-.$$

We give here a lemma regarding generalized almost contact metric structures that will be useful in what follows.

**Lemma 3.2:** Let  $(\Phi, E_{\pm}, G)$  be a (twisted) generalized almost contact metric structure on  $M^{2n+1}$ . Then the following statements hold:

- (i)  $G(E_{\pm}) = E_{\mp}$
- (ii)  $G\Phi = \Phi G$
- (iii)  $G(E^{(1,0)}) = E^{(1,0)}$
- (iv)  $(e^B \Phi e^{-B}, e^B E_{\pm}, e^B G e^{-B})$  is a (twisted) generalized almost contact metric structure, where  $B$  is a B-field transformation. Furthermore, this (twisted) generalized almost contact metric structure is strong if  $(\Phi, E_{\pm})$  is strong.

*Proof.* Since  $(\Phi, E_{\pm}, G)$  is a generalized almost contact metric structure we have

$$0 = -\Phi G \Phi(E_+) = G(E_+) - E_+ \otimes E_+(E_+) - E_- \otimes E_-(E_+) = G(E_+) - E_-$$

and so  $G(E_+) = E_-$ . Similarly one shows  $G(E_-) = E_+$ .

For property (ii), recall we have,

$$-\Phi G \Phi = G - E_+ \otimes E_+ - E_- \otimes E_-.$$

Apply  $\Phi$  to both sides getting

$$-\Phi^2 G \Phi = \Phi G.$$

Using the formula (2.2) for  $\Phi^2$ , we get

$$G\Phi - E_+ \otimes E_- \circ G\Phi - E_- \otimes E_+ \circ G\Phi = \Phi G$$

But Lemma 1 in [11] gives that  $E_{\pm} \circ \Phi = 0$ . This combined with (i) and the self-adjointness of  $G$  then imply

$$G\Phi = \Phi G.$$

To establish property (iii) we show first that  $E^{(1,0)} \subset G(E^{(1,0)})$ . Let  $Y + \beta \in E^{(1,0)}$ . Then  $Y + \beta = X + \alpha - \sqrt{-1}\Phi(X + \alpha)$  for some  $X + \alpha \in TM \oplus T^*M$  such that  $\langle X + \alpha, E_{\pm} \rangle = 0$ . By property (i) and the fact that  $G$  is self-adjoint, we obtain

$$0 = \langle G(X + \alpha), E_{\pm} \rangle.$$



Now consider

$$G(X + \alpha) - \sqrt{-1}\Phi G(X + \alpha) \in E^{(1,0)}.$$

By applying  $G$  again and using the fact that  $G^2 = \text{Id}$  and  $\Phi$  and  $G$  commute gives the first inclusion.

To show inclusion in the other direction, let  $Y + \beta \in G(E^{(0,1)})$ . Then  $Y + \beta = G(X + \alpha - \sqrt{-1}\Phi(X + \alpha))$  for some  $X + \alpha \in TM \oplus T^*M$  such that  $\langle X + \alpha, E_{\pm} \rangle = 0$ . But,

$$0 = \langle X + \alpha, E_{\pm} \rangle = \langle X + \alpha, G(E_{\mp}) \rangle = \langle G(X + \alpha), E_{\mp} \rangle$$

since  $G$  is self-adjoint. Thus,  $G(X + \alpha) - \sqrt{-1}\Phi G(X + \alpha) \in E^{(1,0)}$ . Since  $\Phi$  and  $G$  commute,  $Y + \beta = G(X + \alpha) - \sqrt{-1}\Phi G(X + \alpha) \in E^{(1,0)}$ .

For property (iv), Sekiya [24] showed that  $(e^B\Phi e^{-B}, e^B E_{\pm})$  is again a (twisted) generalized almost contact structure. It remains to show that  $e^B G e^{-B}$  satisfies the compatibility condition 3.1 with sections  $e^B E_{\pm}$ . This reduces to showing that

$$e^B(E_{\pm} \otimes E_{\pm})e^{-B} = e^B E_{\pm} \otimes e^B E_{\pm}.$$

Let  $X + \alpha \in T \oplus T^*$  and so

$$(e^B(E_{+} \otimes E_{+})e^{-B})(X + \alpha) = E_{+}(X + \alpha - \iota_X B)e^B E_{+} = (E_{+}(X + \alpha) - \iota_{\xi_+} \iota_X B)e^B E_{+}$$

where we have used that  $E_{+} = \xi_{+} + \eta_{+}$ . On the other hand, we have

$$\begin{aligned} e^B E_{+} \otimes e^B E_{+}(X + \alpha) &= (e^B E_{+})(X + \alpha)e^B E_{+} = (E_{+} + \iota_{\xi_+} B)(X + \alpha)e^B E_{+} \\ &= (E_{+}(X + \alpha) + \iota_X \iota_{\xi_+} B)e^B E_{+} = (E_{+}(X + \alpha) - \iota_{\xi_+} \iota_X B)e^B E_{+} \end{aligned}$$

where we have used the general property that  $\iota_X \iota_Y = -\iota_Y \iota_X$ . A similar argument is used to show

$$e^B(E_{-} \otimes E_{-})e^{-B} = e^B E_{-} \otimes e^B E_{-}.$$

The strong property follows immediately since the (twisted) Courant bracket is invariant under  $B$ -field transforms. Hence  $L^{\pm}$  being (twisted) Courant involutive is preserved as well. (See also Proposition 3.42 in [14].)  $\square$

**REMARK 3.1:** Observe that an easy consequence of Lemma 3.2 (i) and (ii) together with (3.1) is that  $(M, G\Phi, GE_{\pm} = E_{\mp}, G)$  is again a generalized almost contact metric structure.

Now that all of the pieces are in place, we are ready for our definition of a generalized coKähler structure.

**Definition 3.3:** *A normal generalized contact metric structure  $(M, \Phi, E_{\pm}, G)$  is (H-twisted) generalized coKähler if  $G\Phi$  is also strong with respect to the (H-twisted) Courant bracket.*

**REMARK 3.2:** The sections associated to  $G\Phi$  are  $GE_{\pm} = E_{\mp}$  and so automatically we get that  $[[E_{\pm}, E_{\mp}]] = 0$  for the generalized contact metric structure associated with  $G\Phi$ . Hence, we could have alternatively defined a coKähler structure to be a generalized contact metric structure  $(M, \Phi, E_{\pm}, G)$  such that both  $(M, \Phi, E_{\pm}, G)$  and  $(M, G\Phi, E_{\mp}, G)$  are normal.

If  $(M, \Phi, E_{\pm}, G)$  is a normal generalized contact metric structure then by definition  $\Phi$  is strong. It is important to emphasize that there may be normal generalized contact metric structures where  $G\Phi$  is not strong as the following example shows.

**EXAMPLE 3.4:** Let  $M = SU(2)$ . On the Lie algebra  $su(2)$  choose a basis  $\{X_1, X_2, X_3\}$  and a dual basis  $\{\sigma^1, \sigma^2, \sigma^3\}$  such that  $[X_i, X_j] = -X_k$  and  $d\sigma^i = \sigma^j \wedge \sigma^k$  for cyclic permutations of  $\{i, j, k\}$ . One can construct a classical normal almost contact structure by taking  $\phi = X_2 \otimes \sigma^1 - X_1 \otimes \sigma^2$ ,  $\xi = X_3$  and  $\eta = \sigma^3$ . Then, as in Example 2.13, we can construct a generalized almost contact structure by letting

$$\Phi = \begin{pmatrix} \phi & 0 \\ 0 & -\phi^* \end{pmatrix}, \quad E_+ = X_3, \quad E_- = \sigma^3$$

where  $(\phi^*\alpha)(X) = \alpha(\phi(X))$ ,  $X \in TM$ ,  $\alpha \in T^*M$ . One computes easily that  $E_{\phi}^{(1,0)} = \text{span}\{X_1 - \sqrt{-1}X_2, \sigma^1 - \sqrt{-1}\sigma^2\}$  so that  $L^+ = \text{span}\{X_3, X_1 - \sqrt{-1}X_2, \sigma^1 - \sqrt{-1}\sigma^2\}$  and  $L^- = \text{span}\{\sigma^3, X_1 - \sqrt{-1}X_2, \sigma^1 - \sqrt{-1}\sigma^2\}$ . For  $L^+$ , the relevant Courant brackets give

$$[[X_1 - \sqrt{-1}X_2, \sigma^1 - \sqrt{-1}\sigma^2]] = 0, \quad [[X_3, \sigma^1 - \sqrt{-1}\sigma^2]] = -\sqrt{-1}(\sigma^1 - \sqrt{-1}\sigma^2)$$

as well as  $[[X_3, X_1 - \sqrt{-1}X_2]] = \sqrt{-1}(X_1 - \sqrt{-1}X_2)$ . Similarly, for  $L^-$  the relevant Courant bracket is

$$[[\sigma^3, X_1 - \sqrt{-1}X_2]] = \sqrt{-1}(\sigma^1 - \sqrt{-1}\sigma^2).$$

Since  $(\phi, \xi, \eta)$  is normal, we have that  $[[E_+, E_-]] = \mathcal{L}_{X_3}\sigma^3 = 0$ . Thus  $(\Phi, E_{\pm})$  is a normal generalized contact structure.

Now, define a generalized metric  $G$  on  $TM \oplus T^*M$  by  $\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$  where  $g$  is any Riemannian metric compatible with the almost contact structure. It is a straightforward calculation to verify  $G$  is compatible with the  $(\Phi, E_{\pm})$ . So we have that  $G\Phi$  defines a generalized almost contact structure on  $SU(2)$ . But observe that

$$L_{G\Phi}^+ = \text{span}\{\sigma^3, X_1 - \sqrt{-1}\sigma^2, X_2 + \sqrt{-1}\sigma^1\}$$

and hence  $[[X_1 - \sqrt{-1}\sigma^2, X_2 + \sqrt{-1}\sigma^1]] = -X_3 \notin L_{G\Phi}^+$ . Therefore,  $G\Phi$  is not strong even though  $\Phi$  is strong.

**REMARK 3.3:** If we instead use an H-twisted Courant bracket twisted by the real closed three form  $H = \sigma^1 \wedge \sigma^2 \wedge \sigma^3$  then one can similarly show that  $\Phi$  is strong and  $G\Phi$  is not strong.

Recall that a Kähler structure on  $M$  induces a generalized Kähler structure on  $M$ . The odd dimensional version of this holds as well:

**Proposition 3.5:** *Any coKähler manifold is generalized coKähler.*

*Proof.* Let  $(\phi, \xi, \eta, g)$  be a coKähler structure on  $M$ . Let  $\pi$  be the bivector field as defined in Example 2.14 and let  $\Omega$  be the fundamental two form as given in (2.2). Define

$$\Phi_{\phi} = \begin{pmatrix} \phi & 0 \\ 0 & -\phi^* \end{pmatrix}, \quad \Phi_{\Omega} = \begin{pmatrix} 0 & \pi^{\sharp} \\ \Omega^{\flat} & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}.$$

We will argue that  $(\Phi_{\phi}, E_{\pm}, G)$  is a generalized coKähler structure on  $M$ . Let us first verify that such a  $G$  is compatible with  $(\Phi_{\phi}, E_+ = \xi, E_- = \eta)$ . The compatibility

condition (3.1) for our case reduces to verifying the following equality:

$$(3.2) \quad \phi g^{-1} \phi^* \alpha + \phi^* g \phi X = g^{-1} \alpha + gX - \alpha(\xi)\xi - \eta(X)\eta,$$

where  $X + \alpha \in T \oplus T^*$ . Now,  $\phi^* g \phi X = g(\phi X, \phi) = g(X, \cdot) - \eta(X)\eta$ , using the compatibility of  $g$ . Furthermore, since  $g$  is a Riemannian metric which induces an isomorphism between  $T$  and  $T^*$ , let us write  $\alpha = g(Y)$  for some  $Y \in T$ . Then,

$$\phi g^{-1} \phi^* gY = \phi g^{-1} g(Y, \phi) = -\phi g^{-1} g(\phi Y, \cdot) = -\phi^2 Y.$$

But this is precisely,  $g^{-1} \alpha - \alpha(\xi)\xi$ , using the fact that  $g(Y, \xi) = \eta(Y)$  and  $g(X, \phi Y) = -g(\phi X, Y)$ .

Since we assume  $(\phi, \xi, \eta, g)$  is a coKähler structure on  $M$ , it is normal. Thus,  $[[E_+, E_-]] = [[\xi, \eta]] = 0$ . Moreover, by Proposition 3.4 of [22]  $(M, \Phi_\phi, E_+ = \xi, E_- = \eta)$  is strong. Thus we have showed  $(M, \Phi_\phi, E_+ = \xi, E_- = \eta)$  is a normal generalized contact metric structure. Now, a straightforward computation shows  $G\Phi_\phi = \Phi_\Omega$ . Again, in [22], it was shown  $\Phi_\Omega$  is strong. Therefore,  $(M, \Phi_\phi, E_\pm, G)$  is a generalized coKähler structure.  $\square$

#### 4. PROOF OF THE MAIN THEOREM

Before we begin the proof of Theorem 1.3, it will be useful to formulate a twisted version of Theorem 1.2. Fix the generalized almost contact structures  $(M_i, \Phi_i, E_{\pm, i})$ ,  $i = 1, 2$ . Now on the product even dimensional manifold  $M_1 \times M_2$  we have the generalized almost complex structure given by (1.2) in [11]. We showed in [11] that  $(M_1 \times M_2, \mathcal{J})$  is a generalized complex structure if and only if  $(M_i, \Phi_i, E_{\pm, i})$  are each strong generalized contact structures and  $[[E_{\pm, i}, E_{\mp, i}]] = 0$ . It is easy to see that this theorem trivially extends to the  $H$ -twisted case. The only new ingredient needed in the proof is how the  $\tilde{H}$ -twisted Courant bracket behaves on the product  $M_1 \times M_2$ . It is given by:

$$(4.1) \quad [[(A, B), (C, D)]]_{\tilde{H}} = ([[A, C]]_{H_1}, [[B, D]]_{H_2})$$

where  $H_i$  is a real closed three form on  $M_i$ , ( $i = 1, 2$ ),  $\tilde{H} = H_1 + H_2$ ,  $(A, B)$  and  $(C, D)$  are sections of the generalized tangent bundle of  $M_1 \times M_2$ . Here is now the twisted version of Theorem 1.2.

**Theorem 4.1:** [11] *Let  $M_1$  and  $M_2$  be odd dimensional smooth manifolds each with ( $H_i$ -twisted) generalized almost contact structures  $(\Phi_i, E_{\pm, i})$   $i = 1, 2$ . Then  $M_1 \times M_2$  admits an ( $\tilde{H}$ -twisted) generalized almost complex structure  $\mathcal{J}$ . Further  $\mathcal{J}$  is an ( $\tilde{H}$ -twisted) generalized complex structure if and only if both  $(\Phi_i, E_{\pm, i})$   $i = 1, 2$  are strong ( $H_i$ -twisted) generalized contact structures and  $[[E_{\pm, i}, E_{\mp, i}]]_{H_i} = 0$ .*

To prove Theorem 1.3, we have to introduce generalized metrics  $G_i$  which are compatible with  $\Phi_i$ ,  $i = 1, 2$ . Then on  $M_1 \times M_2$ , with its product metric  $G = (G_1, G_2)$ , we will have two generalized almost complex structures  $\mathcal{J}_1$  and  $\mathcal{J}_2 = G\mathcal{J}_1$  that commute. We will prove that  $\mathcal{J}_2$  is integrable if and only if  $G_i\Phi_i$  are strong.

First we record the following lemma which will be useful. This was proven in [11] in the case in which  $H = 0$  but it is trivial to show that the lemma still holds in the twisted case.

**Lemma 4.2:** [11] *Let  $(M, \Phi, E_{\pm})$  be a ( $H$ -twisted) generalized almost contact structure. Then  $\Phi$  is strong if and only if  $[[L^+, E^{(1,0)}]]_H \subset E^{(1,0)}$  and  $[[L^-, E^{(1,0)}]]_H \subset E^{(1,0)}$ .*

Now we state the main theorem to be proved.

**Theorem 1.3:** *Let  $M_1$  and  $M_2$  be odd dimensional manifolds each with a ( $H_i$ -twisted) generalized coKähler structure  $(\Phi_i, E_{\pm,i}, G_i)$   $i = 1, 2$ . Furthermore, let  $\mathcal{J}_1$  be defined as in (1.2) and let  $\mathcal{J}_2 = G\mathcal{J}_1$  where  $G$  is the product metric. Then  $(M_1 \times M_2, \mathcal{J}_1, \mathcal{J}_2)$  is ( $\tilde{H}$ -twisted) generalized Kähler if and only if  $(\Phi_i, E_{\pm,i}, G_i)$   $i = 1, 2$  are ( $H_i$ -twisted) generalized coKähler.*

*Proof.* Assume  $M_1$  and  $M_2$  are odd dimensional manifolds each with ( $H_i$ -twisted) generalized coKähler structure  $(\Phi_i, E_{\pm,i}, G_i)$ . On  $M_1 \times M_2$  we have the product metric  $G = (G_1, G_2)$ . We know from Theorem 4.1 that  $M_1 \times M_2$  admits a generalized complex structure

$$\mathcal{J}_1(X_1 + \alpha_1, X_2 + \alpha_2) = (\Phi_1(X_1 + \alpha_1) - 2\langle E_{+,2}, X_2 + \alpha_2 \rangle E_{+,1} - 2\langle E_{-,2}, X_2 + \alpha_2 \rangle E_{-,1},$$

$$(4.2) \quad \Phi_2(X_2 + \alpha_2) + 2\langle E_{+,1}, X_1 + \alpha_1 \rangle E_{+,2} + 2\langle E_{-,1}, X_1 + \alpha_1 \rangle E_{-,2}).$$

Thus it is enough to produce a second generalized complex structure that commutes with  $\mathcal{J}_1$ . Note that  $\mathcal{J}_1$  and  $G$  commute by direct computation. So define  $\mathcal{J}_2 = G\mathcal{J}_1$ . Then  $\mathcal{J}_2^2 = -\text{Id}$  and  $\mathcal{J}_1\mathcal{J}_2 = \mathcal{J}_2\mathcal{J}_1$ .

One can now compute an explicit formula for  $\mathcal{J}_2$ :

$$\mathcal{J}_2(X_1 + \alpha_1, X_2 + \alpha_2) = (G\Phi_1(X_1 + \alpha_1) - 2\langle E_{-,2}, X_2 + \alpha_2 \rangle E_{+,1} - 2\langle E_{+,2}, X_2 + \alpha_2 \rangle E_{-,1},$$

$$(4.3) \quad G\Phi_2(X_2 + \alpha_2) + 2\langle E_{-,1}, X_1 + \alpha_1 \rangle E_{+,2} + 2\langle E_{+,1}, X_1 + \alpha_1 \rangle E_{-,2}).$$

A direct calculation shows that  $\mathcal{J}_2^* = -\mathcal{J}_2$ . All that remains to be shown is that the  $\sqrt{-1}$  eigenspaces of  $\mathcal{J}_2$  are closed under the ( $\tilde{H}$ -twisted) Courant bracket.

From the formula for  $\mathcal{J}_2$  we see that the generators of its  $\sqrt{-1}$  eigenspace are given by

$$(E_{G_1\Phi_1}^{(1,0)}, 0)$$

$$(0, E_{G_2\Phi_2}^{(1,0)})$$

$$(E_{+,1}, -\sqrt{-1}E_{+,2})$$

$$(4.4) \quad (E_{-,1}, -\sqrt{-1}E_{-,2}).$$

So it is enough to verify that these generators are closed under the ( $\tilde{H}$ -twisted) Courant bracket. Since  $G_1\Phi_1$  is strong, we have

$$[[E_{G_1\Phi_1}^{(1,0)}, 0], [E_{G_1\Phi_1}^{(1,0)}, 0]]_{\tilde{H}} = ([[E_{G_1\Phi_1}^{(1,0)}, E_{G_1\Phi_1}^{(1,0)}]]_{H_1}, 0) \subset (E_{G_1\Phi_1}^{(1,0)}, 0)$$

by Lemma 4.2 and (4.1).

Similarly,

$$[[E_{G_1\Phi_1}^{(1,0)}, 0], (E_{\pm,1}, -\sqrt{-1}E_{\pm,2})]_{\tilde{H}} = ([[E_{G_1\Phi_1}^{(1,0)}, E_{\pm,1}]]_{H_1}, 0) \subset (E_{G_1\Phi_1}^{(1,0)}, 0)$$

and

$$[[0, E_{G_2\Phi_2}^{(1,0)}], [0, E_{G_2\Phi_2}^{(1,0)}]]_{\tilde{H}} = (0, [[E_{G_2\Phi_2}^{(1,0)}, E_{G_2\Phi_2}^{(1,0)}]]_{H_2}) \subset (0, E_{G_2\Phi_2}^{(1,0)}).$$

Furthermore,

$$[[ (0, E_{G_2\Phi_2}^{(1,0)}), (E_{\pm,1}, -\sqrt{-1}E_{\pm,2}) ]]_{\tilde{H}} = (0, [[ E_{G_2\Phi_2}^{(1,0)}, -\sqrt{-1}E_{\pm,2} ]]_{H_2}) \subset (0, E_{G_2\Phi_2}^{(1,0)}).$$

Since  $[[E_{\pm,i}, E_{\mp,i}]]_{H_i} = 0$ , it is straightforward to compute that

$$[[ (E_{+,1} - \sqrt{-1}E_{+,2}), (E_{-,1}, -\sqrt{-1}E_{-,2}) ]]_{\tilde{H}} = (0, 0)$$

and so the  $\sqrt{-1}$  eigenbundle of  $\mathcal{J}_2$  is Courant closed and thus  $(M_1 \times M_2, \tilde{\mathcal{J}}_1, \mathcal{J}_2, G)$  is generalized Kähler.

Conversely, assume  $M_1 \times M_2$  is a ( $\tilde{H}$ -twisted) generalized Kähler manifold with ( $\tilde{H}$ -twisted) generalized complex structures  $\mathcal{J}_1$  and  $\mathcal{J}_2$  as given above. We must show  $(\Phi_i, E_{\pm,i}, G_i)$  are ( $H_i$ -twisted) generalized coKähler for  $i = 1, 2$ . By applying Theorem 4.1 to  $(M_1 \times M_2, \mathcal{J}_1)$  we get immediately that  $(\Phi_i, E_{\pm,i}, G_i)$  are normal for  $i = 1, 2$ . Since  $\mathcal{J}_2 = G\mathcal{J}_1$  is induced from  $G_i\Phi_i$ , we can apply Theorem 4.1 again to  $(M_1 \times M_2, \mathcal{J}_2)$  and this shows  $G_i\Phi_i$  are normal. Therefore,  $(\Phi_i, E_{\pm,i}, G_i)$  is a ( $\tilde{H}_i$ -twisted) generalized coKähler structure for  $i = 1, 2$ .  $\square$

Here is another proof of Proposition 3.5 as an application of our main theorem.

**Corollary 4.3:** *Any coKähler manifold is generalized coKähler.*

*Proof.* Let  $(\phi, \xi, \eta, g)$  be a coKähler structure on  $M$  and let  $\mathbb{R}$  have its trivial coKähler structure. Now  $M \times \mathbb{R}$  with its product metric is Kähler (see Example 2.3). Therefore it is generalized Kähler. By applying Theorem 1.3, this gives  $(M, \phi, \xi, \eta, g)$  is generalized coKähler.  $\square$

## 5. SOME EXAMPLES OF GENERALIZED COKÄHLER STRUCTURES

We have already seen that every classical coKähler structure gives a generalized coKähler structure. In this section, we provide many more examples of generalized coKähler structures on manifolds. The examples we construct arise from two general constructions: *i*) deformations of generalized Kähler structures and *ii*) products of manifolds.

First, we show that the  $B$ -field transformation of a generalized coKähler structure is again a generalized coKähler structure.

EXAMPLE 5.1: Consider the ( $H$ -twisted) generalized coKähler structure  $(\Phi_\phi, E_\pm, G)$  and let  $B$  be a closed two form,  $\Omega$  the fundamental two form as defined in (2.1), and  $\pi$  the bivector field from Example 2.14. Perform  $B$ -field transformations obtaining

$$\Phi_\phi^B = \begin{pmatrix} \phi & 0 \\ B\phi + \phi^*B & -\phi^* \end{pmatrix}, \Phi_\Omega^B = \begin{pmatrix} -\pi^\sharp B & \pi^\sharp \\ \Omega^\flat - B\pi^\sharp B & B\pi^\sharp \end{pmatrix}$$

and the generalized metric given by

$$G^B = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}.$$

Observe that  $[[e^B\xi, e^B\eta]]_H = [[\xi + B\xi, \eta]]_H = 0$ . Furthermore it can be easily calculated that  $G^B\Phi_\phi^B = \Phi_\Omega^B$ . Since the ( $H$ -twisted) Courant bracket is invariant under  $B$ -field transformations,  $(\Phi_\phi^B, e^B E_\pm, G^B)$  is again ( $H$ -twisted) generalized coKähler.

Recall from Example 2.2 we considered the product of a Kähler manifold  $(N, J, g)$  and  $\mathbb{R}$  (or  $S^1$ .) Using the trivial coKähler structure on  $\mathbb{R}$ , one can construct on  $N \times \mathbb{R}$  a coKähler structure. This product construction can be extended to the generalized context, providing a source of many examples.

**Proposition 5.2:** *Let  $(M, \mathcal{J}_1, \mathcal{J}_2, G_M)$  be a  $(H_M$ -twisted) generalized Kähler manifold and let  $(N, \Phi_1, E_{N,\pm}, G_N)$  be a  $(H_N$ -twisted) generalized coKähler manifold. Then  $M \times N$  admits a  $(H_{M \times N}$ -twisted) generalized coKähler structure.*

*Proof.* Recall that  $T(M \times N) \oplus T^*(M \times N) \approx (TM \oplus T^*M) \oplus (TN \oplus T^*N)$ . Define the endomorphism  $\Phi$  on  $(T^*M \oplus TM) \oplus (TN \oplus T^*N)$  by

$$\Phi = (\mathcal{J}, \Phi_1).$$

Define  $E_+ = (0, E_{N,+})$ , and  $E_- = (0, E_{N,-})$ . Let  $G = G_M \times G_N$  be the product metric. It is easy to verify that  $(\Phi, E_{\pm}, G)$  is a generalized almost contact metric structure on  $M \times N$ . Let  $L$  denote the  $\sqrt{-1}$  eigenbundle of  $\mathcal{J}$ . Then  $L_{\Phi}^{\pm} = (L, E_{\Phi_1}^{(1,0)}) \oplus L_{(0, E_{\pm})}$  is clearly closed under the  $(H_{M \times N}$  twisted) Courant bracket which implies that  $\Phi$  is strong. Also, observe that  $[[E_+, E_-]]_{H_{M \times N}} = 0$ . Hence,  $(\Phi, E_{\pm}, G)$  is a normal generalized contact structure. Similarly  $L_{G\Phi}^{\pm}$  is easily seen to be closed under the  $(H_{M \times N}$  twisted) Courant bracket so  $G\Phi$  is strong. Therefore,  $(M \times N, \Phi, E_{\pm}, G)$  defines a  $(H_{M \times N}$  twisted) generalized coKähler structure.  $\square$

In [12], Goto proves a stability theorem for generalized Kähler structures on a manifold  $M$  under the hypothesis that there exists an analytic family of generalized complex structures on the manifold. Further, Goto shows that the space of obstructions to deformations of generalized complex structures vanishes in the case of a compact Kähler manifold with a holomorphic Poisson structure  $\beta$ . We can use these theorems in combination with the above product theorem to construct examples of nontrivial generalized coKähler manifolds. By *nontrivial generalized coKähler*, we mean that the generalized coKähler structure does not come from a classical coKähler structure or a B-field transform of a classical coKähler structure. We first state Goto's stability theorem:

**Theorem 5.3:** [12] *Let  $M$  be a compact Kähler manifold of dimension  $n$ . If we have an action of an  $l$  dimensional complex commutative Lie group  $G$  with a non-trivial 2-vector  $\beta$ , then we have a family of deformations of nontrivial generalized Kähler structures on  $M$ .*

We combine this theorem with Proposition 5.2 to construct nontrivial generalized coKähler structures.

**EXAMPLE 5.4:** Let  $M$  be any compact toric Kähler manifold so that the hypothesis of Goto's theorem is satisfied. Then  $M$  then admits a nontrivial generalized Kähler structure. Equip  $S^1$  with the trivial generalized coKähler structure given by

$$\Phi = 0, \quad E_+ = \partial_t, \quad E_- = dt, \quad G = \begin{pmatrix} 0 & g_{S^1}^{-1} \\ g_{S^1} & 0 \end{pmatrix}$$

where  $t$  is the coordinate on  $S^1$  and  $g_{S^1} = dt \otimes dt$ . Form the product  $M \times S^1$  which by Proposition 5.2 is generalized coKähler. The nontriviality of the generalized Kähler structure on  $M$  implies the nontriviality of the generalized coKähler structure on  $M \times S^1$ .

It would be interesting to find examples of strictly almost (co)Kähler manifolds that admit a generalized (co)Kähler structure but we were unable to do so except in the twisted case. Gualtieri in [14] showed that the Hopf surface  $S^3 \times S^1$ , which is nonKähler, does not admit any generalized Kähler structure yet it does admit a twisted generalized Kähler structure. For additional examples, see for instance ([1],[16],[12]).

We will construct new examples of almost (co)Kähler non-(co)Kähler manifolds which admit  $H$ -twisted generalized (co)Kähler structures. (A remark on terminology:  $(M, \omega, J, g)$  is strictly almost Kähler if  $M$  does not admit any integrable complex structure and  $(M, \omega, J, g)$  is almost Kähler nonKähler if that particular  $J$  is not integrable.) Our examples begin with a construction done by Fino and Tomassini [9] in which they explicitly construct a six-dimensional solvmanifold which admits an  $H$ -twisted generalized Kähler structure. This manifold, denoted by  $M^6$ , arises as the total space of a  $\mathbb{T}^2$ -bundle over the Inoue surface. From the construction, they are able to compute the first Betti number and show  $b_1(M^6) = 1$ . Therefore, the manifold  $M^6$  is a strictly almost Kähler manifold which admits an  $H$ -twisted generalized Kähler structure.

**EXAMPLE 5.5:** (*Twisted Generalized CoKähler of arbitrary odd dimension  $> 7$* )  
Form the product manifold  $M^7 = M^6 \times S^1$  where  $S^1$  has the trivial ( $H = 0$  twisted) generalized coKähler structure and we get that  $M^7$  admits an  $\tilde{H}$ -twisted generalized coKähler structure by Proposition 5.2. Since  $b_1(M^6) = 1$  [9], it follows that  $b_1(M^7) = 2$ . By Theorem 2.4, this implies  $M^7$  is strictly almost coKähler. The manifold  $M^6$  constructed by Fino and Tomassini actually works in arbitrary even dimension [9] and so denote this manifold by  $M^{2n}$ . Now, form the product with  $S^1$  and define  $M^{2n+1} := M^{2n} \times S^1$ . By Proposition 5.2,  $M^{2n+1}$  admits an  $\tilde{H}$ -twisted generalized coKähler structure and since  $b_1(M^{2n+1}) = 2$ , it is strictly almost coKähler.

The approach we use in the next example follows closely an argument given by Watson [27] in which he constructs higher dimensional almost Kähler non-Kähler manifolds starting with Thurston's torus bundle  $W^4$  over  $T^2$  which has  $b_1(W^4) = 3$ .

**EXAMPLE 5.6:** (*Twisted Generalized Kähler*)  
Proceeding with the manifold  $M^7$  in Example 5.5, form the product  $M^{14} := M^7 \times M^7$ . By Theorem 1.3,  $M^{14}$  admits an  $\tilde{H}_2$ -twisted generalized Kähler structure, where  $\tilde{H}_2 = \tilde{H}_1 + \tilde{H}_1$ . Moreover,  $M^{14}$  with this product structure is an almost Kähler non-Kähler manifold since if it was a Kähler manifold then each factor  $M^7$  would be coKähler by Theorem 1.1, which is impossible since  $b_1(M^7) = 2$ . We can continue this process now. Form the product  $M^{15} := M^{14} \times S^1$ . This manifold is almost coKähler non-coKähler since if it was coKähler then  $M^{14}$  would be Kähler. We can now apply Theorem 1.1 to  $M^{22} := M^{15} \times M^7$  concluding that  $M^{22}$  is an almost Kähler non-Kähler manifold that admits an  $\tilde{H}_3$ -twisted generalized Kähler structure. If  $M^{22}$  were Kähler, then both  $M^{15}$  and  $M^7$  would be coKähler, which cannot happen. At each iteration, one takes the product with an  $S^1$  followed by a product with  $M^7$ . Continuing in this manner, we get  $8n + 6$ -dimensional almost Kähler non-Kähler manifolds which are  $H$ -twisted generalized Kähler and with  $b_1 = 3n + 1$ ,  $n = 0, 1, 2, 3, \dots$ . Note that for  $n = 2k$ , the first Betti number is  $b_1 = 6k + 1$  and so these  $16k + 6$  dimensional manifolds are strictly almost Kähler manifolds which admit twisted generalized Kähler structures. In this procedure we also generate  $8n + 7$ -dimensional almost coKähler non-coKähler manifolds which admit twisted generalized coKähler structures. Moreover, for  $n = 2m$  the  $16m + 7$  dimensional manifolds are strictly almost coKähler manifolds which are

twisted generalized coKähler since  $b_1 = 6m + 2$ . By Theorem 2.4, these manifolds are strictly almost coKähler.

REMARK 5.1: All of these examples are non-diffeomorphic to the examples given by Fino and Tomassini in [9] since our examples have first Betti number which grows linearly with dimension whereas their examples have  $b_1 = 1$  in arbitrary even dimension.

An important feature of generalized Kähler geometry is its relationship with bi-Hermitian geometry. It was shown by Gualtieri in [14] that having a generalized Kähler structure on a manifold is equivalent to having a bi-Hermitian structure on the manifold. Therefore, Example 5.6 gives examples of manifolds which admit bi-Hermitian structures as well.

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