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Renewal Systems, Sharp-Eyed Snakes, And Shifts Of Finite Type

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1. INTRODUCTION. A common theme in all of mathematics is deciding when two seemingly different objects are actually the same in some sense. For example, every elementary school student knows that, despite outward appearances, the fractions $2/3$ and $10/15$ are the same. The calculus student knows that the functions $f(x) = 3x$ and $g(x) = (3x^2 - 3x)/(x - 1)$ are the same, provided we restrict their domains away from one, and the math major knows that the group of symmetries of the square and the dihedral group D_4 are the same group.

In this work we look at collections of bi-infinite strings of symbols that occur in several different areas of mathematics, including coding theory and symbolic dynamical systems, and we ask whether these collections are the same in some sense. In the jargon of the field of symbolic dynamics, we ask a question first posed by Roy Adler: Is the set of renewal systems the same as the set of shifts of finite type?

Of course, the statement that one mathematical object "is the same as" another mathematical object has different meanings in different circumstances. Fractions are the same if they are identical when put in lowest terms. Two functions on a domain are the same if every input yields the same output. Algebraic groups are the same if there is a one-to-one correspondence between group elements that preserves the structure of the group operation, that is, if the groups are isomorphic. Part of our task will be to investigate what it means for the mathematical objects that we are interested in—namely, renewal systems and shifts of finite type—to be the same. Once we've agreed on what it *means* for these objects to be the same, we will try to determine whether they actually *are* the same. We will provide a partial answer to Adler's question leaving us with a tantalizing open problem that is the subject of active mathematical research.

2. RENEWAL SYSTEMS AND SHIFTS OF FINITE TYPE. A *digraph* consists of a set of vertices and a set of directed edges between vertices. An example of a digraph with three vertices and seven edges is shown in Figure 1. We will require that all digraphs be *strongly connected*; that is, given any two vertices, there is a sequence of edges starting at one vertex and ending at the other.

A *loop digraph* (or *flower digraph*) is a digraph with the additional property that each vertex, except possibly one, has a single incoming edge. The vertex with more than one incoming edge we refer to as the *central vertex*. (If every vertex has a single incoming edge, any vertex may be designated the central vertex.) Figure 2 depicts an example of a loop digraph.

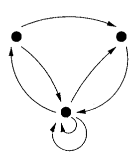

Figure 1.

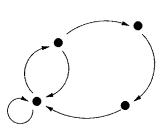

Figure 2.

Fix $\mathcal{A}$, a finite collection of symbols that we call an *alphabet*. Typical choices are a set of positive integers (e.g., $\mathcal{A} = \{1, 2, \ldots, n\}$) or a set of English letters (e.g., $\mathcal{A} = \{a, b, c\}$). With $\mathcal{A}$ now designated, a *labeled digraph* is a digraph with its edges labeled by symbols from $\mathcal{A}$. Two possible labelings of the digraph in Figure 1 that use the alphabet $\mathcal{A} = \{1, 2, 3, 4, 5, 6, 7\}$ are presented in Figure 3.

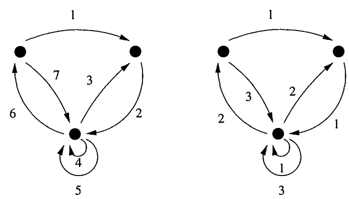

Figure 3.

Note that each edge may have a distinct label, as in the first digraph in Figure 3, or several edges may share the same label, as in the second digraph in Figure 3. For reasons that will become clear, a labeled digraph whose edges are labeled distinctly will be referred to as an *SFT-digraph* and will be denoted by $\mathcal{G}$.

Figure 4 shows a possible labeling of the loop digraph from Figure 2. We will denote a labeled loop digraph by $\mathcal{LG}$.

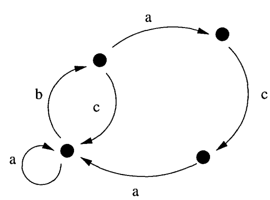

Figure 4.

Let $\mathcal{A}^Z$ denote the set of all bi-infinite sequences of symbols from $\mathcal{A}$. A labeled digraph determines a subset of $\mathcal{A}^Z$: imagine a bug standing on an edge in a labeled digraph and hopping from edge to edge following the allowed directions on the edges. At each edge on which the bug lands, it reads off the label on that edge. If the bug has done this for an infinite number of past hops and will continue for an infinite number of future hops, the labels that it reads off as it moves through this path correspond to a bi-infinite sequence in $\mathcal{A}^Z$.

We can now define the two objects that will be of interest to us in this paper:

Definition 2.1. A *renewal system* is the set of bi-infinite sequences that can be obtained from a labeled loop digraph $\mathcal{LG}$. We will denote a renewal system by $X_{\mathcal{LG}}$.

Renewal systems are also known to coding theorists as *loop systems* or *flower automata* [**1**].

Definition 2.2. A *shift of finite type* is the set of bi-infinite sequences that can be obtained from an SFT-digraph $\mathcal{G}$. We will denote a shift of finite type by $X_{\mathcal{G}}$.

There are other definitions of a shift of finite type, for instance, one using matrices and another using a set of forbidden blocks. Using the notion of "sameness" that we will develop, these various definitions are the same. We refer the interested reader to [**7**]. Notice that the set of bi-infinite sequences on a "loop SFT-digraph" is both a renewal system and a shift of finite type.

The following sequence comes from the digraph shown in Figure 4:

$$\ldots a \quad b \quad c \quad .b \quad a \quad c \quad a \ldots$$

The period signifies the "time zero" position; i.e., the bug hopping through this path is currently on the edge labeled b, the edge whose label is immediately to the right of the period. Symbols to the right of that represent the future. The bug will hop from its current position on the edge labeled b to the adjacent edge labeled a, then to the edge labeled c, and so on. Similarly, the edges to the left of the period represent the past. We use the following notation to denote the passage of time:

$$\sigma(\ldots a \quad b \quad c \quad .b \quad a \quad c \quad a \ldots) = \ldots a \quad b \quad c \quad b \quad .a \quad c \quad a \ldots$$

Thus, for a bi-infinite sequence x, $\sigma(x)$ represents the same bi-infinite sequence with "time zero" moved one step into the future. We call σ the *shift map*.

Shifts of finite type and renewal systems are studied by mathematicians in a variety of different contexts. They are examples of *symbolic dynamical systems*. Many systems, such as the motion of the planets or of gas molecules, can be modeled by a space (the possible states) and a transformation of that space (the passage of time). By dividing the space into a finite number of pieces, each associated with a symbol, we obtain a symbolic dynamical system. A bi-infinite sequence of symbols represents the "pieces" of the state space occupied by a particular planet or molecule at fixed time intervals extending into the infinite past and infinite future. Many qualitative aspects of the original system can be explored in this way.

The term "symbolic dynamics" was first used by Hedlund and Morse [**9**] in 1938, but the idea of a symbolic dynamical system can be found as early as 1898, when Hadamard described all the bi-infinite sequences that can occur in modeling geodesic flows on surfaces of negative curvature [**3**]. For more on symbolic dynamical systems, see Lind and Marcus [**7**] or Kitchens [**5**].

Another area of mathematics where shifts of finite type and renewal systems turn up is coding and information theory. Data is not stored or transmitted verbatim, but as long sequences of symbols. The data storage scheme used must satisfy various constraints imposed by the physical limitations of the hardware used. Many sets of bi-infinite sequences constructed to satisfy these constraints, such as those constructed with the runlength limited system method that is currently in use in many disc drives, are shifts of finite type [**7**].

Coding theorists are also interested in codes that store data efficiently and codes in which it is easy to detect and correct errors that might have occurred in the course of transmitting data. Finite sequences of symbols, called *blocks*, are used to store data and a *code* is a finite collection of allowable blocks. The set of bi-infinite sequences constructed using this collection is clearly a renewal system. (For more on coding and information theory, see Berstel and Perrin [**1**], Hamming [**4**], or Lidl and Pilz [**6**].)

Of obvious importance to anyone studying these systems is the question of what it should mean to say that two such systems are the same. Once that is established, the next question becomes: When are two given systems the same? In Section 3 we will explore what it means for a renewal system and a shift of finite type to be the same. Then in Section 4, with our definition of "the same" in hand, we ask whether the collection of renewal systems is the same as the collection of shifts of finite type.

3. CONJUGATE SYSTEMS. The set of bi-infinite sequences from a loop SFT-digraph is both a renewal system and a shift of finite type. But certainly not all renewal systems are shifts of finite type and not all shifts of finite type are renewal systems, as the following examples demonstrate.

Example 3.1. *The set of bi-infinite sequences on the digraph $\mathcal{LG}$ in Figure 5 is a renewal system that is not a shift of finite type [**11**].*

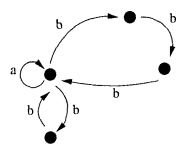

Figure 5.

It is important to point out that, when arguing that the renewal system of Example 3.1 is not a shift of finite type, it is not enough to say that the loop digraph in Figure 5 is not an SFT-digraph. Instead, we have to argue that there is *no* SFT-digraph that would give the same collection of bi-infinite sequences as the one given by the loop digraph in Figure 5. However, a little thought should convince one that this is true, for any SFT-digraph giving the same bi-infinite sequences as the loop digraph in Figure 5 would have distinctly labeled edges. Thus, it would have exactly two edges, one labeled a and one labeled b, so there are only two possibilities for our SFT-digraph. These possibilities are illustrated by Figure 6.

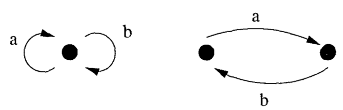

Figure 6.

Both of these digraphs allow for the bi-infinite sequence

$$\dots a \quad b \quad a \quad b \quad .a \quad b \quad a \quad b \quad a \quad b \quad a \quad b \dots,$$

which is not a possible bi-infinite sequence for the digraph $\mathcal{LG}$. Thus this renewal system cannot be a shift of finite type. ■

Example 3.2. *The set of bi-infinite sequences on the digraph $\mathcal{G}$ in Figure 7 is a shift of finite type that is not a renewal system.*

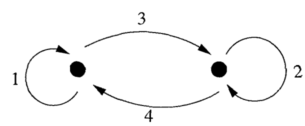

Figure 7.

As with Example 3.1, we must argue that there is no loop digraph giving the same set of bi-infinite sequences as the SFT-digraph in Figure 7. We note that one of the bi-infinite sequences we obtain from the digraph in Figure 7 is $\ldots 111111.111111\ldots$. If a loop digraph gives the same set of bi-infinite sequences, it must have a sequence of edges starting and ending at the central vertex all of which are labeled 1. Similarly, it must have a sequence of edges starting and ending at the central vertex all of which are labeled 2. But then the bi-infinite sequence $\ldots 222.111\ldots$ would occur in the renewal system and yet it is clearly not in the shift of finite type. So there is no loop digraph that will give the same set of bi-infinite sequences as the SFT-digraph in Figure 7. ■

While they do intersect, the collection of renewal systems and the collection of shifts of finite type are not the same. Examples 3.1 and 3.2 show that there are renewal systems that are not shifts of finite type, and vice versa. However, in making this statement we are being quite literal: we regard a renewal system and a shift of finite type as the same if they consist of identical sequences.

This literal interpretation of "sameness" is obviously too restrictive. Consider the shift of finite type $X_{\mathcal{G}}$ determined by the digraph on the left in Figure 8 and the renewal system $X_{\mathcal{LG}}$ determined by the digraph on the right. These can not be the same, since the bi-infinite sequences in $X_{\mathcal{G}}$ consist of the symbols 1 and 2, whereas the bi-infinite sequences in $X_{\mathcal{LG}}$ consist of the symbols a and b. Yet it is obvious that a simple renaming of the edges on the digraph $\mathcal{LG}$ would yield a renewal system that is the same as $X_{\mathcal{G}}$. Certainly any notion of "sameness" that we use should equate these two systems.

Figure 8.

By merely renaming symbols, we obtain a correspondence between bi-infinite sequences that preserves the nature of the sequences as completely as possible. For g, a bi-infinite sequence in $X_{\mathcal{G}}$ corresponding to γ in $X_{\mathcal{LG}}$, the shifted sequence $\sigma(g)$ corresponds to the shifted image of g, that is, to $\sigma(\gamma)$. For example, when renaming 1 to a and 2 to b, the sequence

$$g = \ldots 1\ \ 2\ \ 2\ \ 2\ \ .1\ \ 2\ \ 1\ \ 2\ \ 1 \ldots$$

in $X_{\mathcal{G}}$ corresponds to

$$\gamma = \ldots a\ \ b\ \ b\ \ b\ \ .a\ \ b\ \ a\ \ b\ \ a \ldots$$

in $X_{\mathcal{LG}}$. Using the same renaming rule, the shifted bi-infinite sequence

$$\sigma(g) = \ldots 1\ \ 2\ \ 2\ \ 2\ \ 1\ \ .2\ \ 1\ \ 2\ \ 1 \ldots$$

corresponds to

$$\ldots a\ \ b\ \ b\ \ b\ \ a\ \ .b\ \ a\ \ b\ \ a \ldots,$$

which is $\sigma(\gamma)$, the shifted image of g.

Any notion of sameness should have this property, since two bi-infinite sequences that correspond should continue to do so when the analogous shift of "time zero" is made in each. We are thus led to the following definition:

Definition 3.3. Suppose f maps bi-infinite sequences in the shift of finite type $X_{\mathcal{G}}$ to bi-infinite sequences in the renewal system $X_{\mathcal{LG}}$ in such a way that $f(\sigma(x)) = \sigma(f(x))$ for any sequence x from $X_{\mathcal{G}}$. Then we say that f *commutes with the shift map*.

Intuitively, a mapping between bi-infinite sequences that commutes with the shift map "preserves time."

There is an additional property that is important in deciding whether a renewal system and a shift of finite type are the same. We will want any correspondence between sequences to "preserve ongoing trends." This means roughly that two bi-infinite sequences in $X_{\mathcal{G}}$ that are the same for long periods of time should have images in $X_{\mathcal{LG}}$ that are the same for long periods of time.

An example that does not "preserve ongoing trends" is easy to describe. Consider again the shift of finite type and renewal system whose digraphs are shown in Figure 8. Suppose we decide to map the bi-infinite sequence of all 1's in $X_{\mathcal{G}}$ to the bi-infinite sequence of all b's in $X_{\mathcal{LG}}$ and the bi-infinite sequence of all 2's in $X_{\mathcal{G}}$ to the bi-infinite sequence of all a's in $X_{\mathcal{LG}}$. To map the nonconstant bi-infinite sequences in $X_{\mathcal{G}}$ to bi-infinite sequences in $X_{\mathcal{LG}}$, change the 1's to a's and the 2's to b's. Such a map f surely commutes with the shift map. It is also one-to-one and onto. However, it doesn't "preserve ongoing trends." To see this, consider the bi-infinite sequences $x_n = \ldots 2222111\ldots 11.11\ldots 1112222\ldots$ in $X_{\mathcal{G}}$, where there are $2n$ 1's in the center of x_n. Under our mapping, this will correspond to $f(x_n) = \ldots bbbbaaa\ldots aa.aa\ldots aaabbbb\ldots$ in $X_{\mathcal{LG}}$, a bi-infinite sequence with $2n$ a's in the center. As n increases, x_n agrees with the bi-infinite sequence of all 1's for longer and longer center sections, but $f(x_n)$ does not agree with $f(\ldots 1111.1111\ldots) = \ldots bbbb.bbbb\ldots$ for longer and longer center sections. So f does not "preserve ongoing trends."

Definition 3.4. A correspondence f between the bi-infinite sequences of $X_{\mathcal{G}}$ and the bi-infinite sequences of $X_{\mathcal{LG}}$ *preserves ongoing trends* if for every $M > 0$ there exists $N > 0$ so that if sequences x and y agree on the N symbols centered at the time zero position, then $f(x)$ and $f(y)$ agree on the M symbols centered at the time zero position.

(The observant reader might recognize the requirement that f preserve ongoing trends as a continuity requirement. The topology of the alphabet $\mathcal{A}$ is the discrete topology and the topology of $X_{\mathcal{G}}$ (or $X_{\mathcal{LG}}$) is the product topology. The mapping f must be continuous with respect to this topology.)

Definition 3.5. A shift of finite type $X_{\mathcal{G}}$ and a renewal system $X_{\mathcal{LG}}$ are the "same" if there is a one-to-one, onto correspondence $f : X_{\mathcal{G}} \to X_{\mathcal{LG}}$ between their bi-infinite sequences that commutes with the shift map and preserves ongoing trends. In this case we say that the shift of finite type and the renewal system are *conjugate*.

The terminology given in Definition 3.5 is used in the field of symbolic dynamical systems. When two systems are conjugate, they are considered to be the same symbolic dynamical system. In coding theory, a correspondence f as in Definition 3.3 is said to be *time invariant* and is called a *feedback free encoder* [**8**].

Renaming symbols gives an exceptionally simple correspondence. Are there other examples of systems that are conjugate but where the correspondence is a bit more involved? Let us explore this issue in the following three examples.

Example 3.6. *A shift of finite type that is conjugate to a renewal system using a snake.*

Consider once more the shift of finite type X_G from Example 3.2 and the renewal system given by the digraph $\mathcal{LG}$ on the left in Figure 6. A bug walking along a path in digraph $\mathcal{G}$ from Figure 7 could construct a corresponding path on digraph $\mathcal{LG}$ by merely reading out a when it was on edge 1 or edge 4 and reading out b when it was on edge 2 or edge 3. The mapping this procedure generates certainly commutes with the shift map and preserves ongoing trends, and a little thought convinces us that it is onto. But is it one-to-one?

A common way to check that a mapping is one-to-one is to find its inverse. That is, as the bug hops along a path γ in $\mathcal{LG}$, can it construct the path g in $\mathcal{G}$ that would correspond to γ under the mapping in the preceding paragraph? This will not be as easy. When the bug is on edge a in digraph $\mathcal{LG}$, it doesn't know whether that should correspond to a 1 or a 4 in digraph $\mathcal{G}$ unless it knows from whence it has just come. If the preceding symbol were a b, which corresponds to a 2 or 3, then the a following it would have to correspond to a 4. If the preceding symbol were an a, which corresponds to a 1 or 4, then the a following it would have to correspond to a 1. Unfortunately, a bug hopping from edge to edge has no memory of where it has just been.

Suppose instead that we have a snake slithering along the digraph $\mathcal{LG}$ and reading off the symbol on the edge its head lies on. The length of the snake is the length of two edges on the digraph. The snake's tail lies on the edge that its head has just passed over, so in that respect, its tail contains a record of where it has just been. Because the snake knows where it has just been, it is able to construct the inverse mapping to the one described above. The snake's four possible positions on $\mathcal{LG}$ (taking into account the position of its tail as well as its head) correspond to the four edges on $\mathcal{G}$ (Figure 9). As it slithers along a bi-infinite sequence on digraph $\mathcal{LG}$, it can construct a corresponding bi-infinite sequence on digraph $\mathcal{G}$.

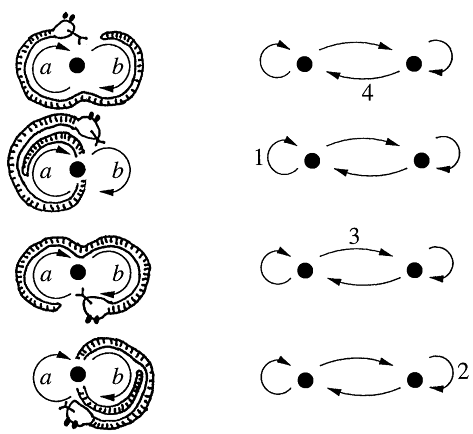

Figure 9.

As an example, consider the following bi-infinite sequence g obtained from the SFT-digraph of Example 3.2:

$$g = \ldots 4\ 3\ 4\ 3\ .2\ 2\ 4\ 1\ 3 \ldots.$$

Replacing 1 and 4 with a and 2 and 3 with b gives the corresponding bi-infinite sequence γ on the pertinent loop digraph in Figure 6:

$$\gamma = \ldots a\ b\ a\ b\ .b\ b\ a\ a\ b \ldots.$$

Then the correspondence given by the snake's position as it moves along this bi-infinite sequence in digraph $\mathcal{LG}$ shown in Figure 7 returns the initial sequence

$$g = \ldots 3\ 4\ 3\ .2\ 2\ 4\ 1\ 3 \ldots$$

on digraph $\mathcal{G}$.

These two mappings are inverses of each other and thus both are one-to-one and onto. It is clear that they both commute with the shift map and preserve ongoing trends. As a consequence, the shift of finite type in Example 3.2 is conjugate to the renewal system determined by the loop digraph that appears on the left in Figure 6. ■

Example 3.7. *A shift of finite type that is conjugate to a renewal system using a longer snake.*

The snake in the previous example was two edge lengths long. We note that at times a longer snake may be required. Consider the digraphs $\mathcal{G}$ (left) and $\mathcal{LG}$ (right) in Figure 10.

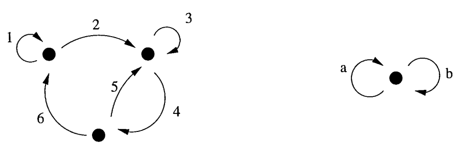

Figure 10.

We will first define a correspondence between bi-infinite sequences in $X_{\mathcal{G}}$ and bi-infinite sequences in $X_{\mathcal{LG}}$ by assigning 1, 4, and 6 to a, and the remaining symbols in $\mathcal{G}$ to b. We leave it to the reader to convince herself that a two-edge-length snake will not be able to find the inverse of this correspondence. However, a snake that is three edge lengths long could construct an inverse using the correspondences in the following chart. (An asterisk indicates that either an a or b may appear in this location.) ■

Snake end	Mid snake	Snake head	Corresponds to symbol:
a	a	a	1
a	a	b	2
$*$	b	b	3
$*$	b	a	4
b	a	b	5
b	a	a	6

Example 3.8. *A shift of finite type that is conjugate to a renewal system using a sharp-eyed snake* [**11**].

Consider the renewal system given in Example 3.1 and the shift of finite type given by the SFT-digraph $\mathcal{G}$ in Figure 11.

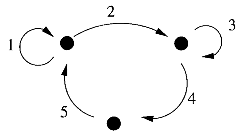

Figure 11.

A bug hopping along a bi-infinite path in digraph $\mathcal{G}$ could construct a corresponding bi-infinite path on digraph $\mathcal{LG}$ in Figure 5 merely by reading out a when it was on edge 1 or 5 and reading out b when it was on any other edge. Thus, for example, the bi-infinite sequence $g = \ldots 3452.33345112\ldots$ in digraph $\mathcal{G}$ will correspond to the bi-infinite sequence $\gamma = \ldots bbab.bbbbaaab\ldots$ in $\mathcal{LG}$. This mapping would certainly commute with the shift map and preserve ongoing trends, and a moment's thought should convince one that it takes the bi-infinite sequences in $X_{\mathcal{G}}$ onto the bi-infinite sequences in $X_{\mathcal{LG}}$. Again we ask: Is the mapping one-to-one? Is there an inverse mapping taking bi-infinite sequences in $X_{\mathcal{LG}}$ to bi-infinite sequences in $X_{\mathcal{G}}$?

Our two-edge-length snake cannot answer this one. If the snake were slithering along a path γ in $X_{\mathcal{LG}}$ with its head on a and its tail on b, that should certainly correspond to its head being on the symbol 5 in the corresponding bi-infinite sequence g in $X_{\mathcal{G}}$. Similarly, if its head were on b and its tail on a, that should correspond to its head being on the symbol 2 in the corresponding bi-infinite sequence g in $X_{\mathcal{G}}$. However, suppose that both its head and tail were on b. Should that correspond to its head being on 3 or 4? This is unclear. Giving the snake a longer tail so it can remember even more of the past does not help, since for any length string $bbb\ldots bb$ occurring in some path in $\mathcal{LG}$, there are strings $333\ldots 34$ and $333\ldots 33$ occurring in paths in $\mathcal{G}$ that correspond to $bbb\ldots bb$.

However, knowing a bit about the future will overcome this problem. Suppose that our snake has sharp eyes, giving it a limited view of the future. If the snake is slithering along a path γ in $X_{\mathcal{LG}}$ with its tail and its head on b, sharp eyesight would allow it to see if it will be moving to a b; in that case, certainly the b that its head is on should correspond to its head being on the symbol 3 in the corresponding path g in $X_{\mathcal{G}}$. Similarly, if its tail and head are on b, and it can see that it will be moving to an a, then the symbol b that its head is on should correspond to its head being on the symbol 4 in the corresponding path g in $X_{\mathcal{G}}$.

Snake end	Snake head	Snake sees	Corresponds to symbol:
a	a	$*$	1
a	b	b	2
b	b	b	3
b	b	a	4
b	a	$*$	5

Continuing in this way, the snake determines a correspondence between a triple of symbols in a bi-infinite sequence γ in $X_{\mathcal{L}\mathcal{G}}$ to a symbol in the corresponding bi-infinite sequence g in $X_{\mathcal{G}}$. As the snake moves along a bi-infinite sequence γ in $X_{\mathcal{L}\mathcal{G}}$, it observes the symbols under its head and tail as well as the symbol on the upcoming edge, and it then uses this correspondence to determine where its head would be at the same time on the corresponding bi-infinite sequence g in $X_{\mathcal{G}}$. For example, the bi-infinite sequence $\gamma = \ldots abbbb.bbbaab\ldots$ in $X_{\mathcal{L}\mathcal{G}}$ corresponds to the bi-infinite sequence $g = \ldots 2333.33451\ldots$ in $X_{\mathcal{G}}$. ∎

It turns out that any mapping that commutes with the shift map and preserves ongoing trends as in Definition 3.5 can always be described using a snake with a long enough tail and sharp enough eyesight. The term *sliding block code* is used in symbolic dynamical systems for such a mapping. (The authors wish to credit Daniel Ullman for the idea of using a snake to visualize a sliding block code. See Ullman [**10**].)

Notice that when we use Definition 3.5 for "sameness," every example of a shift of finite type considered thus far has been conjugate to a renewal system, and every example of a renewal system has been conjugate to a shift of finite type. So, again we ask: Is the collection of renewal systems the same, in the sense of Definition 3.5, as the collection of shifts of finite type? In the next section we discuss what is known, and what is still unknown, about the answer to this question.

4. ARE SHARP-EYED SNAKES ENOUGH? We have seen that the set of renewal systems and the set of shifts of finite type do intersect. First of all, any loop SFT-digraph gives both a renewal system and a shift of finite type. More generally, there are renewal systems that are the same as, or to use the more technical term, conjugate to, shifts of finite type; conversely, there are shifts of finite type that are conjugate to renewal systems. It is natural to ask whether this is always the case: Is every renewal system conjugate to a shift of finite type? Is every shift of finite type conjugate to a renewal system? We will see that while the members of a certain class of renewal systems are known to be conjugate to shifts of finite type, in fact not every renewal system is conjugate to a shift of finite type. Interestingly, whether or not every shift of finite type is conjugate to a renewal system is unknown, and we will close by discussing this current, active research question.

A Class of Renewal Systems Conjugate to Shifts of Finite Type. Consider the renewal system given by the labeled loop digraph in Figure 12.

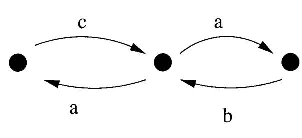

Figure 12.

A possible bi-infinite sequence for this digraph is

$$\ldots a\ \ b\ \ a\ \ b\ \ .a\ \ c\ \ a\ \ b\ \ a\ \ c\ \ a\ \ c\ldots.$$

Notice that the bi-infinite sequence above corresponds to a unique path on the digraph, for the symbols ac must correspond to the left loop on the digraph and ab must correspond to the right loop. (By a "loop" we mean a path that starts and ends at the central vertex, but does not otherwise pass through the central vertex.) We can divide this bi-infinite sequence of symbols according to the loops on the digraph in this unique path:

$$\ldots \underline{a\ b}\ \underline{a\ b}\ .\underline{a\ c}\ \underline{a\ b}\ \underline{a\ c}\ \underline{a\ c} \ldots.$$

In fact, any bi-infinite sequence in this renewal system will correspond to a unique path since it must consist entirely of ac's or ab's, each of which can arise from only one loop on the digraph. This is not always the case. For instance, consider the renewal system whose loop digraph is illustrated in Figure 13.

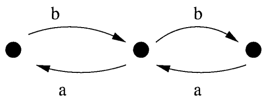

Figure 13.

This renewal system contains the bi-infinite sequence

$$\ldots a\ b\ a\ b\ .a\ b\ a\ b\ a\ b\ a\ b \ldots,$$

which can arise from two different paths on the digraph, the one that repeats the left loop, the loop labeled ab, infinitely many times, or the one that repeats the right loop, the loop labeled ba, infinitely many times.

Renewal systems like the one associated with Figure 12 are called *uniquely decipherable*, a term arising in coding theory.

Definition 4.1. A renewal system is *uniquely decipherable* if every bi-infinite sequence obtained from the defining digraph corresponds to a unique path on the digraph.

The reader can check that the renewal systems whose defining digraphs are found in Figures 4 and 6 are uniquely decipherable, whereas the renewal system defined by the digraph in Figure 5 is not.

Theorem 4.2. *A uniquely decipherable renewal system is conjugate to a shift of finite type.*

Before proving Theorem 4.2, we make an observation. In the uniquely decipherable example whose defining digraph is shown in Figure 12, it was easy to see how to divide a bi-infinite sequence of symbols into blocks that correspond to walking around loops on the digraph; in that way, every symbol in a bi-infinite sequence can be associated uniquely with an edge in the digraph. When an a appears in a bi-infinite sequence, the symbol to the right of the a determines the corresponding edge in the digraph.

However, even if we know that a bi-infinite sequence of symbols corresponds to a unique path on the digraph, determining the corresponding edge of a particular symbol in the bi-infinite sequence is more difficult for more complicated digraphs. Consider, for instance, the renewal system whose loop digraph is given in Figure 4. This digraph is depicted again in Figure 14, with its edges relabeled for clarity.

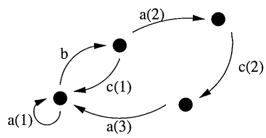

Figure 14.

The symbol b corresponds to a unique edge but a and c do not. To determine the edge corresponding to a particular symbol a, we must look to its past. If it is preceded by a or bc, it corresponds to $a(1)$. If it is preceded by b, then it corresponds to $a(2)$; if by bac, then to $a(3)$. If the symbol c is preceded by b, it corresponds to $c(1)$ and if preceded by a, to $c(2)$. Clearly with more complicated digraphs consisting of long loops and lots of repeated symbols, determining the unique corresponding path would be even more difficult.

It turns out that, for every uniquely decipherable system, there is an integer M that tells us how many symbols to the right and left are needed in order to determine which edge in which loop corresponds to a given symbol. (The existence of M requires a compactness argument, where the topology of the renewal system is the product topology described earlier: if no such M exists, then a limit point argument establishes the existence of two distinct paths corresponding to the same bi-infinite sequence of symbols.)

We are now ready for the proof of Theorem 4.2.

Proof. Let $X_{\mathcal{LG}}$ be the uniquely decipherable renewal system associated with a labeled loop digraph $\mathcal{LG}$. Let k be the number of loops in the digraph $\mathcal{LG}$ and call these loops $\omega_1, \omega_2, \cdots, \omega_k$. For instance, in the example in Figure 4, $\omega_1 = a$, $\omega_2 = bc$, and $\omega_3 = baca$. To prove the theorem, we will first describe a certain shift of finite type and then prove that it is conjugate to our renewal system.

Our shift of finite type will be defined with an SFT-digraph $\mathcal{G}$, a loop digraph with k loops. The jth edge in the ith loop will be labeled $\omega_i(j)$. For example, the digraph we would construct for the renewal system whose defining digraph is shown in Figure 4 is illustrated in Figure 15. The shift of finite type $X_{\mathcal{G}}$ that we intend to prove is conjugate to our original renewal system will be all bi-infinite sequences obtained from the SFT-digraph constructed in this way.

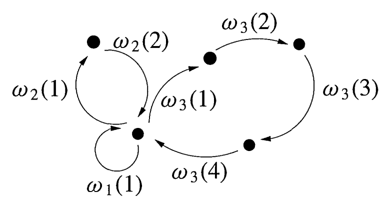

Figure 15.

To show that $X_{\mathcal{G}}$ is conjugate to $X_{\mathcal{LG}}$, we need to find a correspondence of the type described in Definition 3.5 between their bi-infinite sequences. The correspondence is easy to describe: given a symbol $\omega_i(j)$ in a bi-infinite sequence in $X_{\mathcal{G}}$, replace it with the label on the jth edge of loop ω_i in $\mathcal{LG}$. This replaces a bi-infinite sequence in $X_{\mathcal{G}}$ with a bi-infinite sequence in $X_{\mathcal{LG}}$. It is clear that this correspondence is onto, commutes with the shift map, and preserves ongoing trends, but is it one-to-one? Can we find an inverse? Since our renewal system is uniquely decipherable, there is an integer M such that looking forward M positions and backwards M positions around a symbol in a bi-infinite sequence in $X_{\mathcal{LG}}$ is enough to determine the particular edge on the particular loop that corresponds to the given symbol. We can then replace the symbol in the bi-infinite sequence in the renewal system with the symbol $\omega_i(j)$ in the unique corresponding bi-infinite sequence in the shift of finite type. One can check that this is the inverse of the above map, and thus the two systems are conjugate. ■

If a renewal system $X_{\mathcal{LG}}$ is uniquely decipherable, then it is conjugate to a shift of finite type. If $X_{\mathcal{LG}}$ is not uniquely decipherable, it may still be conjugate to a shift of finite type, as we have seen in Example 3.8. So the question remains: Are all renewal systems, including the nonuniquely decipherable ones, conjugate to shifts of finite type? The next theorem shows that the answer to this question is no.

A Renewal System that Is not Conjugate to a Shift of Finite Type.

Theorem 4.3. *There exists a nonuniquely decipherable renewal system that is not conjugate to a shift of finite type.*

Proof. Consider the renewal system given by the loop digraph $\mathcal{LG}$ in Figure 16.

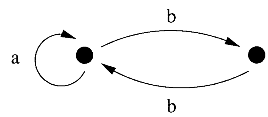

Figure 16.

In symbolic dynamics, this is called the "even system" because, for any bi-infinite sequence for this system, the symbol a must occur between an even number of b's. We will demonstrate that it is not conjugate to any shift of finite type. The proof is indirect. Assume there *is* a conjugacy between $X_{\mathcal{LG}}$ and some shift of finite type $X_{\mathcal{G}}$, say given by a map f. Thus f takes bi-infinite sequences from $X_{\mathcal{LG}}$ to bi-infinite sequences in $X_{\mathcal{G}}$ and has an inverse f^{-1}. From our earlier discussion, we can think of f and f^{-1} as determined by a sharp-eyed snake on the digraphs in question. Let us say that the length of the snake and the keenness of its sight for both f and f^{-1} is M. (There is no loss of generality in making this assumption, for if a snake of length k can determine the mapping, then a longer snake with better eyesight can do so as well.)

For a bi-infinite sequence x, let x_i denote the ith symbol in x when counting from the time zero position, with the symbol immediately to the right of the time zero position being x_0. Consider two bi-infinite sequences x and y from $X_{\mathcal{LG}}$. The sequence y will have $y_i = a$, except that

$$y_{-(2M+1)} = y_{-(2M)} = \cdots = y_{-1} = y_0 = y_1 \cdots = y_{2M+2} = b.$$

The sequence x will be $\sigma(y)$. For example, if $M = 1$, we would have

$$y = \ldots\ a\ \ a\ \ b\ \ b\ \ b\ \ .b\ \ b\ \ b\ \ b\ \ b\ \ a \ldots$$

and

$$x = \ldots\ a\ \ b\ \ b\ \ b\ \ b\ \ .b\ \ b\ \ b\ \ b\ \ a\ \ a \ldots.$$

For simplicity we continue the argument for $M = 1$; it will be clear how this extends to a general M.

Let $\xi = f(x)$ and $\upsilon = f(y)$. Because the snake associated with the map f has a tail that goes back one symbol and eyesight that sees forward one symbol and because $x_{-3}x_{-2}x_{-1}.x_0x_1x_2x_3 = y_{-3}y_{-2}y_{-1}.y_0y_1y_2y_3$, the symbols $\xi_{-2}\xi_{-1}.\xi_0\xi_1\xi_2$ are the same as the symbols $\upsilon_{-2}\upsilon_{-1}.\upsilon_0\upsilon_1\upsilon_2$. Recall that an SFT-digraph has unique labels, so this means the path in $\mathcal{G}$ given by ξ and the one given by υ must be at the same locations at these times. Thus the bi-infinite sequence of symbols created by using the past of ξ and the future of υ is also a path in the $X_{\mathcal{G}}$. Denote this sequence by β. Then $\beta = \cdots \xi_{-2}\xi_{-1}.\upsilon_0\upsilon_1\upsilon_2 \cdots$ is in $X_{\mathcal{G}}$. Now apply f^{-1} to this new bi-infinite sequence. Since the snake associated with f^{-1} also has a tail that goes back one symbol and eyesight that sees forward one symbol and since the symbols up to β_2 are the same as for the bi-infinite sequence ξ, this inverse image has $\cdots x_{-5}x_{-4}x_{-3}x_{-2}x_{-1}$ to the left of the time zero position. Since the symbols starting at β_{-2} are the same as those in the bi-infinite sequence υ, the inverse image will then have $y_0y_1y_2\cdots$ to the right of the time zero position. Thus

$$f^{-1}(\beta) = \ldots \; a \; b \; b \; b \; b \; .b \; b \; b \; b \; b \; a \ldots,$$

which is not a point in $X_{\mathcal{LG}}$. We thereby reach a contradiction. Thus no such conjugacy f exists. ■

Is Every Shift of Finite Type Conjugate to a Renewal System? The answer to this question is unknown. In fact, even for shifts of finite type arising from seemingly simple SFT-digraphs, like the two in Figure 17, it is not known whether there exists a conjugate renewal system.

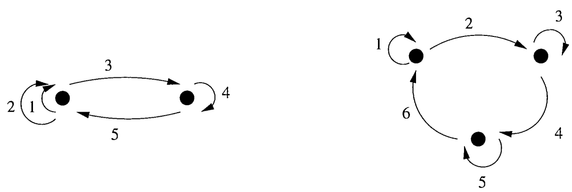

Figure 17.

A dynamical systems property called *entropy* can be used to show that the shifts of finite type are not all conjugate to uniquely decipherable renewal systems. The entropy of a shift of finite type or a renewal system is a number that is related to the growth rate of the number of blocks occurring in bi-infinite sequences as the block size increases. Conjugate systems must have the same entropy, although the converse of this statement is not true. In [2], it is shown that the collection of entropies of uniquely decipherable renewal systems does not include all the possible entropies of shifts of finite type; thus, there are shifts of finite type that cannot be conjugate to a uniquely decipherable renewal system. However, the same paper establishes the fact that for every shift of finite type there is a renewal system with the same entropy. This leaves open the possibility that, if the nonuniquely decipherable renewal systems are included, every shift of finite type may be conjugate to a renewal system.

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